

CONFORMAL WALKER METRICS AND LINEAR FEFFERMAN-GRAHAM EQUATIONS

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ABSTRACT. The conformal Fefferman-Graham ambient metric construction is one of the most fundamental constructions in conformal geometry. It embeds a manifold with a conformal structure into a pseudo-Riemannian manifold whose Ricci tensor vanishes up to a certain order along the original manifold. Despite the general existence result of such ambient metrics by Fefferman and Graham, not many examples of conformal structures with Ricci-flat ambient metrics are known. Motivated by previous examples, for which the Fefferman-Graham equations for the ambient metric to be Ricci-flat reduce to a system of linear PDEs, in the present article we develop a method to find ambient metrics for conformal classes of metrics with two-step nilpotent Schouten tensor. Using this method, for metrics for which the image of the Schouten tensor is invariant under parallel transport, i.e., certain types of Walker metrics, we obtain explicit ambient metrics. This includes certain left-invariant Walker metrics as well as pp-waves.

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1. INTRODUCTION AND MAIN RESULTS

This paper is a follow-up of our papers [12, 13, 1], where we presented several examples of pseudo-Riemannian conformal structures, not conformally Einstein, with explicit Ricci-flat Fefferman-Graham ambient metrics. The Fefferman-Graham ambient metric is a fundamental construction from conformal geometry which is defined as follows:

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Given a conformal class represented by a metric g on a smooth manifold M , a *Fefferman-Graham ambient metric* or just an *ambient metric* is a metric

$$(1.1) \quad \tilde{g} = 2 \, \text{dtd}(\rho t) + t^2(g(x^i) + h(x^i, \rho)),$$

defined on $\tilde{M} = (0, \infty) \times M \times (-\epsilon, \epsilon)$, with coordinates x^i on M , $t \in (0, \infty)$ and $\rho \in (-\epsilon, \epsilon)$, such that $h(x^i, \rho)|_{\rho=0} = 0$ and

$$(1.2) \quad \text{Ric}(\tilde{g}) = O(\rho^m), \quad \text{with } m = \infty \text{ if } n \text{ is odd and } m = \frac{n-2}{2} \text{ if } n \text{ is even.}$$

Fefferman and Graham [4, 5] have shown that an ambient metric always exists and is unique in a certain sense, which justifies it to call it *the* ambient metric. Moreover, when n is even, there is a conformally covariant, divergence and trace free $(0, 2)$ -tensor \mathcal{O} , the *Fefferman-Graham obstruction tensor*, which vanishes whenever (1.2) holds also for $m \geq \frac{n}{2}$.

We will refer to the equations (1.2) for a metric of the form (1.1) as the *Fefferman-Graham equations*. Sometimes we will say that a solution of (1.2) is given by h , by which we mean that h defines a metric \tilde{g} via the formula (1.1) such that $\text{Ric}(\tilde{g}) = O(\rho^m)$. Moreover if equation (1.2) holds for all m when n is even, we emphasise this by call \tilde{g} a *Ricci-flat ambient metric*.

Finding explicit (Ricci-fat) ambient metrics amounts to solving a system of second order PDEs for the unknown symmetric ρ -dependent $(0, 2)$ -tensor field h . In general, these PDE are nonlinear in h , however, for the examples presented in [12, 13, 1], we were able to solve these PDEs explicitly, by the following approach: We found an ansatz for h such that the operator $\text{Ric}(\tilde{g})$ became linear in h , which allowed us to solve the equation $\text{Ric}(\tilde{g}) = 0$. This raises the immediate question: what are the features of the conformal class responsible for this phenomenon? In the present paper we will identify one of these features as a property of the conformal holonomy:

Theorem 1.1. *Let $(M, [g])$ be a conformal manifold such that the conformal holonomy admits an invariant subspace that is totally null and of dimension greater than 1. Then there is a metric g in the conformal class defining the linear differential operator \mathcal{A} acting on ρ -dependent symmetric bilinear forms,*

$$(1.3) \quad \mathcal{A}_{ij}(h) = 2\rho\ddot{h}_{ij} + (2-n)\dot{h}_{ij} + 2R^k_{ij}{}^l h_{kl} - \nabla^k \nabla_k h_{ij},$$

where R_{ijkl} is the curvature tensor of g and the dot denotes the derivative with respect to ρ , such that a solution of equation (1.2) is given via (1.1) by a divergence free symmetric bilinear form h that solves the equation

$$(1.4) \quad h^{kl} \nabla_k \nabla_l h_{ij} + \nabla_k h_{li} \nabla^l h^k_j + \mathcal{A}_{ij}(h) + 2R_{ij} = O(\rho^m),$$

with $m = \infty$ if n is odd and $m = \frac{n-2}{2}$ if n is even, and where R_{ij} is the Ricci tensor of g .

Although the appearance of the quadratic terms in equation (1.4) is somewhat unsatisfactory, in many cases there is an obvious ansatz for h such that the quadratic terms vanish and the resulting linear equation for h can be solved explicitly. We will come back to this.

Recall that the conformal holonomy is defined as follows: to a conformal class of signature (p, q) one can assign the normal conformal $\mathfrak{so}(p+1, q+1)$ -valued Cartan connection which induces a principal connection and in turn a metric connection on the vector bundle

of conformal standard tractors. The conformal holonomy is the holonomy of this connection.

It was shown in a series of papers [9, 14, 15] that the assumption in Theorem 1.1 — that the conformal holonomy admits an invariant totally null subspace of dimension $k + 1 > 1$ — is equivalent to the existence, locally and outside a singular set, of a totally null distribution \mathcal{N} of rank k and a metric g in the conformal class, such that:

- (A) The image of the Schouten tensor P of g is contained in \mathcal{N} (which implies $P^2 = 0$),
- (B) \mathcal{N} is parallel (with respect to the Levi-Civita connection of g).

Metrics with a parallel totally null distribution \mathcal{N} are called *Walker metrics* [16]. Metrics with properties (A) and (B) are special Walker metrics, for which the image of the Schouten tensor, and consequently of the Ricci tensor, is contained in the parallel null distribution \mathcal{N} . In the following we will call metrics that have both properties (A) and (B) *null Ricci Walker metrics*, referring to the property that image of the Ricci tensor is totally null. The case $k = 1$ was considered in [14], where the metrics were called *pure radiation metrics with parallel rays*. There are many known examples of null Ricci Walker metrics. This includes Lorentzian pp-waves but also the examples of metrics we gave in [1], which are of signature $(3, 3)$ and lie in Bryant's conformal classes [3]. Recently, in [7] the ambient metric for *Patterson-Walker metrics* was computed. Patterson-Walker metrics are null Ricci Walker metrics in neutral signature (n, n) that arise from projective structures in dimension n . In Section 5 we will give more examples of null Ricci Walker metrics including left-invariant metrics.

With the above characterisation of the assumption, Theorem 1.1 is a consequence of several results we will prove in this paper. To explain these results, we recall that all the examples in [1] satisfy property (A). This observation combined with the fact that $\partial_\rho h|_{\rho=0} = 2P$, suggested our ansatz for h as a tensor satisfying $\text{Im}(h) \subset \mathcal{N}$. If not only (A) but also (B) is satisfied, which is the case for most but not all of the examples in [1], then we can show that the condition $\text{Im}(h) \subset \mathcal{N}$ is necessary:

Theorem 1.2. *Let (M, g) be a pseudo-Riemannian null Ricci Walker metric with parallel null distribution \mathcal{N} . Then for every ambient metric $\tilde{g} = 2 \text{dtd}(\rho t) + t^2(g(x^i) + h(x^i, \rho))$, i.e., a solution for the equations (1.2), it holds*

$$(1.5) \quad \text{div}^g(h) = O(\rho^m), \quad \text{Im}(h) \subset \mathcal{N} \quad \text{mod } O(\rho^m)$$

with $m = \infty$ when n is odd and $m = \frac{n}{2}$ when n is even. Moreover, when n is even, the obstruction tensor satisfies $\text{Im}(\mathcal{O}) \subset \mathcal{N}$, and there is an ambient metric for which h satisfies equations (1.5) for $m = \infty$.

We will prove this theorem in Section 2.4. Note that the statement about the obstruction tensor can also be obtained from results in [11]. Theorem 1.2 leads us to study the equation (1.2) for \tilde{g} as in (1.1) defined by a Walker metric g with parallel null distribution \mathcal{N} and with a tensor h with $\text{Im}(h) \subset \mathcal{N}$. From the results and computations in Section 3 and Section 4.2 we obtain:

Theorem 1.3. *Let (M, g) be a null Ricci Walker metric with parallel null distribution \mathcal{N} and assume that h is a divergence-free symmetric $(0, 2)$ -tensor field such that $\text{Im}(h) \subset \mathcal{N}$. Then the metric \tilde{g} defined by h via equation (1.1) satisfies (1.2) if and only if h satisfies equation (1.4).*

In the case when the parallel null distribution \mathcal{N} has rank one or satisfies an additional condition on the curvature, we can strengthen this result in the sense that the quadratic terms in equation (1.4) vanish:

Corollary 1.1. *Let $(M, [g])$ be a conformal manifold given by a null Ricci Walker metric g with parallel null distribution \mathcal{N} that has rank one, or satisfies $\mathcal{N} \lrcorner R = 0$, for R the curvature tensor of g . Then there is an ambient metric, i.e., a solution of (1.2), that is given via (1.1) by a divergence free symmetric bilinear form h that solves the linear system of PDEs*

$$(1.6) \quad \mathcal{A}_{ij}(h) + 2R_{ij} = O(\rho^m), \quad \text{with } m = \infty \text{ if } n \text{ is odd and } m = \frac{n-2}{2} \text{ if } n \text{ is even,}$$

where $\mathcal{A}_{ij}(h)$ is the linear differential operator defined in (1.3) and R_{ij} is the Ricci tensor of g . When n is even, the obstruction tensor satisfies $\text{Im}(\mathcal{O}) \subset \mathcal{N}$ and $\mathcal{N} \lrcorner \nabla \mathcal{O} = 0$.

Solving the linear equation (1.6) is of course more feasible than solving the original equation (1.2), and in some cases allows to find explicit Ricci-flat ambient metrics. Before we show these examples, a few remarks are in place:

In Section 3, which is very technical and contains the computation of $\text{Ric}(\tilde{g})$ in terms of h for \tilde{g} as in (1.1), we are able to show that the Fefferman-Graham equations become quadratic in the perturbation h for a *much larger class of metrics* than just the null Ricci Walker metrics. For the equations to become quadratic in h it suffices to assume that

- (1) the image of \mathbf{P} is contained in a totally null distribution \mathcal{N} ,
- (2) and that \mathcal{N}^\perp is *involutive* (but not necessarily parallel).

The form of the Fefferman-Graham equations in this more general situation, although being at most quadratic in h , is however more complicated than equations (1.6). Nevertheless, we find this more general class noteworthy: the examples of \mathbf{G}_2 conformal classes in [13, 1], for which the linear Fefferman-Graham equation were reduced to linear PDEs, are *not* null Ricci Walker metrics but rather from this more general class. The reduction was possible because these metrics satisfy the additional property

$$(1.7) \quad \mathcal{L}_X \mathbf{P} = 0, \quad \text{for all } X \in \mathcal{N},$$

where \mathcal{L}_X denotes the Lie derivative in direction X , which suggested, in addition to (1.5), the simple ansatz

$$(1.8) \quad \mathcal{L}_X h = 0, \quad \text{for all } X \in \mathcal{N}.$$

It turns out that (1.8) is crucial for the vanishing of the quadratic terms in equation (1.4).

The null Ricci Walker metrics that satisfy the additional conditions of Corollary 1.1 also satisfy condition (1.7), and based on this we are able to show in Proposition 4.4 that for such metrics equation (1.8) is not an ansatz but has to be satisfied for the h that defines an ambient metric. The property (1.8) causes the quadratic terms in equations (1.4) to vanish, so that the equations indeed become linear. In Section 5, we are able to construct explicit ambient metrics for a large class of such null Ricci Walker metrics, including left-invariant metrics on Lie groups and generalised pp-waves. Our main results in Section 5 are the following:

Theorem 1.4. *Let \mathfrak{k} be a two-step nilpotent Lie algebra, H be a Lie group with Lie algebra \mathfrak{h} , and $\phi : \mathfrak{h} \rightarrow \mathfrak{der}(\mathfrak{k})$ a Lie algebra homomorphism into the derivations of \mathfrak{k} . Let G be the Lie group corresponding to the Lie algebra \mathfrak{g} that is given as the semi-direct sum*

$$\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{k}.$$

Moreover, let g be a pseudo-Riemannian left-invariant metric on G such that $\mathfrak{z}^{\perp} = \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{h}^{\perp} \oplus \mathfrak{z}$, where \mathfrak{z} is the centre of \mathfrak{k} . Then the conformal class of g on G admits Ricci-flat ambient metrics given by

$$(1.9) \quad \tilde{g} = 2d(\rho t)dt + t^2 \left(g + \frac{2\rho}{n-2} \text{Ric}(g) \right),$$

where n is the dimension of G and $\text{Ric}(g)$ is the Ricci tensor of g .

We should point out that the ambient metric in (1.9) is not unique (when n even or when non-analytic ambient metrics are allowed). In fact, in Theorem 5.1 we find the most general form for Ricci flat ambient metrics for the left-invariant metrics in Theorem 1.4 and show that the ambiguity is parametrised by $\frac{p(p+1)}{2}$ functions of $n-p$ variables, where p is the dimension of \mathfrak{h} .

Finally, amongst other results, in Section 5 we extend our results in [12]:

Theorem 1.5. *Let*

$$g = 2dudv + H du^2 + \sum_{i=1}^{n-2} (dx^i)^2$$

be a Lorentzian pp-wave metric with $H = H(x^1, \dots, x^{n-2}, u)$ a function not depending on v . Let Δ is the flat Laplacian in $n-2$ dimensions. Then an ambient metric for $[g]$, i.e. a solution of (1.2), is given by

$$(1.10) \quad \tilde{g} = 2d(\rho t)dt + t^2 g + t^2 \left(\sum_{k=1}^m \frac{\Delta^k(H)}{k! \prod_{i=1}^k (2i-n)} \rho^k + \sum_{k=0}^{\infty} \frac{\Delta^k(f)}{k! \prod_{i=1}^k (2i+n)} \rho^{\frac{n}{2}+k} \right) du^2,$$

where $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even, and $f = f(x^1, \dots, x^{n-2}, u)$ is an arbitrary smooth function. Moreover:

- (1) When n is odd, $f \equiv 0$ gives the unique Ricci-flat ambient metric that is analytic in ρ .
- (2) When n is even, the obstruction tensor \mathcal{O} is given by

$$\mathcal{O} = c \Delta^{\frac{n}{2}}(H) du^2,$$

for some non-zero constant c . If \mathcal{O} vanishes, the metric (1.10) is Ricci-flat.

In addition to this, we obtain non-analytic Ricci-flat ambient metrics with $h \downarrow 0$ if $\rho \rightarrow 0$ from formulas (5.9) and (5.11) in Theorems 5.2 and 5.3, in particular in the case when n is even and the obstruction tensor does not vanish.

We believe that the formulas we provide in this paper are useful to obtain explicit solutions to the Fefferman-Graham equations for other examples than the ones given in Theorems 1.4 and 1.5.

2. THE FEFFERMAN-GRAHAM AMBIENT CONSTRUCTION

2.1. Fefferman-Graham metric. A *conformal structure* $(M, [g])$ on an $n = n_+ + n_-$ -dimensional manifold M is an equivalence $[g]$ class of (n_+, n_-) -signature metrics on M , such that two metrics g and \hat{g} are in the same class $[g]$ if and only if there exists a function ϕ on M , such that $\hat{g} = e^{2\phi}g$.

Let us focus on a given conformal structure $(M, [g])$. In the following definition of an ambient metric we will refer to a manifold \widetilde{M} that is a product

$$\widetilde{M} = (0, \infty) \times M \times (-\epsilon, \epsilon), \quad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ) .

Definition 2.1. An *ambient metric* \widetilde{g} for $(M, [g])$ (that is in normal form with respect to g) is a metric on \widetilde{M} given by

$$(2.1) \quad \widetilde{g} = 2dt d(\rho t) + t^2 g(x^i, \rho),$$

with a 1-parameter family of symmetric bilinear forms $g(x^i, \rho)$ on M , parametrized by ρ , such that

$$g(x^i, \rho)|_{\rho=0} = g(x^i),$$

for some metric $g = g(x^i)$ from the conformal structure $[g]$ and such that

- $Ric(\widetilde{g}) = O(\rho^\infty)$ if n is odd, and
- $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}-1})$ and $\text{tr}_g \left(\rho^{1-\frac{n}{2}} Ric(\widetilde{g})|_{TM \otimes TM} \right) = 0$ along $\rho = 0$, if n is even.

The existence and uniqueness result for ambient metrics in [4, 5] states that for each choice of $g = g(x^i)$ there is an ambient metric w.r.t. g . In all dimensions $n \geq 3$, $g(x^i, \rho)$ has an expansion of the form

$$g(x^i, \rho) = \sum_{k \geq 0} g^{(k)}(x^i) \rho^k$$

starting with

$$g(x^i, \rho) = g(x^i) + 2\rho P(x^i) + O(\rho^2),$$

where $P = \frac{1}{n-2}(Ric - \frac{Scal}{2(n-1)}g)$ is the Schouten tensor of $g = g(x^i)$. In odd dimensions the Ricci flatness condition determines $g^{(k)}$ uniquely for all k , whereas in even dimensions only the $g^{(k < \frac{n}{2})}$ and the trace of $g^{(\frac{n}{2})}$ are determined uniquely. The ambient metric construction is conformally invariant in the sense that ambient metrics for different metrics in the conformal class are diffeomorphic to each other (modulo $O(\rho^{\frac{n}{2}})$ when n is even).

For n even a conformally invariant $(0, 2)$ -tensor on M , the *ambient obstruction tensor* \mathcal{O} , obstructs the existence of smooth solutions to $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}})$. For \widetilde{g} in normal form w.r.t. g as in Definition 2.1 it is given by

$$(2.2) \quad \mathcal{O} = c_n \left(\rho^{1-\frac{n}{2}} (Ric(\widetilde{g})|_{TM \otimes TM}) \right) |_{\rho=0},$$

where c_n is some known nonzero constant [5]. From this one can deduce that \mathcal{O} is trace- and divergence free.

Remark 2.1. If $[g]$ contains the *flat* metric g_0 than the corresponding ambient metric is

$$\tilde{g} = 2dt d(\rho t) + t^2 g_0.$$

Similarly, if $[g]$ contains an *Einstein* metric g_Λ , $Ric(g_\Lambda) = \Lambda g_\Lambda$, then

$$\tilde{g} = 2dt d(\rho t) + t^2 \left(1 + \frac{\Lambda \rho}{2(n-1)}\right)^2 g_\Lambda.$$

In case when $[g]$ does not contain an Einstein, or even if one has a metric g conformal to Einstein but not Einstein itself, finding an explicit form of the corresponding ambient metric \tilde{g} is difficult.

2.2. The Fefferman-Graham equations. Given a conformal structure and having its representative $\overset{0}{g}$, the search for a corresponding Fefferman-Graham ambient metric

$$\tilde{g} = 2d(\rho t)dt + t^2 g(x, \rho),$$

consists in finding a 1-parameter family $g(x, \rho)$ of metrics on M such that the Ricci tensor of the metric \tilde{g} satisfies equations (1.2). In Ref. [5, Eq. 3.17] the components of $Ric(\tilde{g})$ for (2.1) were written explicitly for the unknown tensor $g = g(x^i, \rho)$. Writing g as $g = g_{ij} dx^i dx^j$, with $g_{ij} = g_{ij}(x^k, \rho)$, equation (1.2) then read as:

$$(2.3) \quad \rho \ddot{g}_{ij} - \rho g^{kl} \dot{g}_{ik} \dot{g}_{jl} + \frac{1}{2} \rho g^{kl} \dot{g}_{kl} \dot{g}_{ij} - \left(\frac{n}{2} - 1\right) \dot{g}_{ij} - \frac{1}{2} g^{kl} \dot{g}_{kl} g_{ij} + R_{ij} = O(\rho^m),$$

$$(2.4) \quad g^{kl} (\nabla_k \dot{g}_{il} - \nabla_i \dot{g}_{kl}) = O(\rho^m),$$

$$(2.5) \quad g^{kl} \ddot{g}_{kl} + \frac{1}{2} g^{kl} g^{pq} \dot{g}_{pk} \dot{g}_{ql} = O(\rho^m),$$

for $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even. Here for each ρ , ∇ is the Levi-Civita connection of the metric $g(x^k, \rho) = g_{ij}(x^k, \rho) dx^i dx^j$, R_{ij} is the Ricci tensor of $g(x^i, \rho)$, and the dot denotes partial derivative of g_{ij} with respect to ρ . The left-hand sides of these equations are the components of the Ricci-tensor $Ric(\tilde{g})$ of \tilde{g} .

The first of the Fefferman-Graham equations above is a system of nonlinear 2nd order PDEs for the coefficients g_{ij} . It is also obvious that finding the general solution for this system with a given initial condition $g_{ij}|_{\rho=0} = \overset{0}{g}_{ij}$ is rather hopeless. One can search for Fefferman-Graham metrics assuming that the metric $g(x, \rho)$ admits a power series expansion with integer powers in ρ . Fefferman and Graham [5] gave expressions for the first few terms in the power series expansion in ρ of $g(x, \rho)$ so that \tilde{g} is Ricci flat up to the order 3. Up to this order, their expansion reads:

$$g = \overset{0}{g} + 2P\rho + \mu\rho^2 + \dots,$$

with P being the Schouten tensor for $\overset{0}{g}$, and

$$(4-n)\mu_{ij} = B_{ij} + (4-n)P_i^k P_{kj}.$$

Here B is the Bach tensor of the metric $\overset{0}{g}$ defined by

$$B_{ij} = \overset{0}{\nabla}^k A_{ijk} - P^{kl} W_{kijl},$$

with

$$A_{ijk} = \overset{0}{\nabla}_j P_{ki} - \overset{0}{\nabla}_k P_{ji}$$

the Cotton tensor. The symbol $\overset{0}{\nabla}$ denotes the Levi-Civita connection for $\overset{0}{g}$ and W^i_{jkl} is the Weyl tensor for g .

2.3. Our approach. Our approach in this paper will be the following: We will write the unknown family of semi-Riemannian metrics $g(x^i, \rho)$ in the Fefferman-Graham metric as

$$g(x^i, \rho) = \overset{0}{g}(x^i) + h(x^i, \rho),$$

where $\overset{0}{g} = \overset{0}{g}(x^i)$ is a suitable metric from the conformal class (independent of ρ) and $h = h(x^i, \rho)$ is symmetric, ρ -dependent symmetric bilinear form on M . For our approach we will express the Levi-Civita connection and the Ricci tensor of $g(x^i, \rho)$, which is needed in equations (2.3, 2.4, 2.5), in terms of the Levi-Civita connection and the Ricci tensor of $\overset{0}{g}$. For this, recall the formulas relating the Levi-Civita connections and the curvatures of two given metrics g_{ij} and $\overset{0}{g}_{ij}$. The difference of both Levi-Civita connections is given by a tensor field C^k_{ij} ,

$$(2.6) \quad \nabla_i X_j - \overset{0}{\nabla}_i X_j = C^k_{ij} X_k,$$

where X_k is a one-form. For vector fields we have

$$\nabla_i X^j - \overset{0}{\nabla}_i X^j = -C^j_{ik} X^k.$$

Since both connections are torsion-free, it is $C^k_{ij} = C^k_{ji}$, which, together with $\nabla_i g_{jk} = 0$, implies

$$(2.7) \quad C^k_{ij} = \frac{1}{2} g^{kl} \left(\overset{0}{\nabla}_l g_{ij} - \overset{0}{\nabla}_i g_{jl} - \overset{0}{\nabla}_j g_{il} \right)$$

For the curvature tensors, defined by $R_{ijk}{}^l v_l = 2\nabla_{[i} \nabla_{j]} v_k$ we obtain

$$R_{ijk}{}^l = \overset{0}{R}_{ijk}{}^l + 2\overset{0}{\nabla}_{[i} C^l_{j]k} + 2C^p_{k[i} C^l_{j]p},$$

and hence for the Ricci tensor

$$(2.8) \quad R_{ij} = R_{ikj}{}^k = \overset{0}{R}_{ij} + \overset{0}{\nabla}_i C^k_{kj} - \overset{0}{\nabla}_k C^k_{ij} + C^p_{ij} C^k_{kp} - C^p_{jk} C^k_{ip}.$$

We will use these formulas later on.

In the following we will also consider symmetric $(0, 2)$ -tensors that are symmetric with respect to the metric $\overset{0}{g}$ and moreover 2-step nilpotent. This will be our assumption on the Schouten tensor \mathbf{P} as well the ansatz for h in the ambient metric. About such tensors, we recall the following algebraic fact:

Lemma 2.1. *Let $(M, \overset{0}{g})$ be a semi-Riemannian manifold of dimension n and $h \in \text{End}(TM)$ be a selfadjoint, i.e., symmetric with respect to $\overset{0}{g}$, endomorphism field such that $h^2 = 0$. Then there is a totally null vector distribution \mathcal{N} such that $\text{Im}(h) \subset \mathcal{N}$ and $\text{Ker}(h) = \text{Im}(h)^\perp \subset \text{Ker}(h)$. More precisely, there exist a local co-frame $\Theta^1, \dots, \Theta^n$, such that*

$$\overset{0}{g} = 2 \sum_{i=1}^p \epsilon_i \Theta^{2i-1} \circ \Theta^{2i} + \sum_{j=2p+1}^n \epsilon_j (\Theta^j)^2,$$

where p is the dimension of the image of h and with $\epsilon_i = \pm 1$, and

$$h^b := \overset{0}{g}(h.,.) = \sum_{i=1}^p \epsilon_i (\Theta^{2i})^2.$$

This lemma follows from a result about the normal form of a linear map h that is selfadjoint with respect to non-degenerate bilinear form $\overset{0}{g}$, see for example [8, Theorem 12.2]. For $h^2 = 0$ this result implies that at a point in M , there is a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, such that h and $\overset{0}{g}$ are given as

$$h = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overset{0}{g} = \begin{pmatrix} \epsilon_1 G & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \epsilon_p G & 0 \\ 0 & 0 & 0 & \mathbf{1}_{(\epsilon_{2p+1}, \dots, \epsilon_n)} \end{pmatrix},$$

in which h consists of $p = \dim(\text{Im}(h))$ Jordan blocks $J := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $G := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\mathbf{1}_{(\epsilon_{2p+1}, \dots, \epsilon_n)}$ is the diagonal matrix with $\epsilon_{2p+1}, \dots, \epsilon_n$ on the diagonal. From this we get that $h^b = \overset{0}{g}(h.,.)$ is given as $h^b(\mathbf{e}_{2i}, \mathbf{e}_{2j}) = \epsilon_i \delta_{ij}$ and zero otherwise. Hence, in the dual frame Θ^i defined by $\Theta^i(\mathbf{e}_j) = \delta^i_j$, h^b and $\overset{0}{g}$ are given as in the lemma. It implies

Corollary 2.1. *Let $(M, \overset{0}{g})$ be a semi-Riemannian manifold with Ricci tensor Ric and Schouten tensor \mathbf{P} . Then the following are equivalent:*

- (1) $\text{Ric}^2 = 0$,
- (2) $\mathbf{P}^2 = 0$,
- (3) $\text{Im}(\mathbf{P})$ is totally null,
- (4) $\text{Im}(\text{Ric})$ is totally null.

If any of these conditions is satisfied, then $(M, \overset{0}{g})$ has vanishing scalar curvature.

2.4. Necessary conditions for the ambient metric of null Ricci Walker metrics.

In this section we will derive conditions on the h of the ambient metric for a conformal class that contains a null Ricci Walker metric $\overset{0}{g}$. Recall that we defined a *null Ricci Walker-manifold*, as a pseudo-Riemannian manifold $(M, \overset{0}{g})$ that admits a vector distribution $\mathcal{N} \subset TM$ of rank $p > 0$ such that \mathcal{N} is totally null with respect to $\overset{0}{g}$, invariant under parallel transport with respect to the Levi-Civita connection $\overset{0}{\nabla}$ of $\overset{0}{g}$, and contains the image of the Schouten tensor \mathbf{P} , or equivalently of the Ricci tensor. Based on the fact that $\dot{h}|_{\rho=0} = 2\mathbf{P}$, our ansatz for h in the following section will be to assume that the image of h is also contained in \mathcal{N} . We will now show that for null Ricci Walker metrics this ansatz is in fact necessary, at least up to the critical order when n is even. The following theorem will imply Theorem 1.2 from the introduction.

Theorem 2.1. *Let $(M, \overset{0}{g})$ be a null Ricci Walker metric of dimension $n > 2$ with Schouten tensor \mathbf{P} whose image is contained in a $\overset{0}{\nabla}$ -parallel totally null distribution \mathcal{N} . Let $\tilde{g} =$*

$2dt\,d(\rho t) + t^2g$ with $g = g(x^i, \rho)$ be an ambient metric for $\overset{0}{g}$ in the sense of Definition 2.1. Then for

$$h = g - \overset{0}{g} = \sum_{m \geq 1} \frac{1}{m!} \overset{m}{h} \rho^m$$

with $\overset{m}{h} = \overset{m}{h}(x^i)$ it holds the following:

(1) If n is odd, then

$$(2.9) \quad \text{Im } \overset{m}{h} \subset \mathcal{N},$$

$$(2.10) \quad \overset{0}{\nabla}_k \overset{m}{h}_i{}^k = 0,$$

for all $m \geq 1$.

(2) If n is even, then (2.9) and (2.10) must hold for $m \leq \frac{n}{2} - 1$ and the obstruction tensor satisfies

$$\text{Im}(\mathcal{O}) \subset \mathcal{N}.$$

Moreover, one can choose an ambient metric such that the corresponding $\overset{m}{h}$ satisfy (2.9) and (2.10) for all $m \geq 1$.

Remark 2.2. The statement about the obstruction tensor in the case n even can also be obtained from results in [11].

Remark 2.3. Note that (2.9) is equivalent to $\overset{m}{h}_{ij} = 0$ unless $i, j \in \{n - p + 1, \dots, n\}$. Moreover, we use the following convention: g^{kl} refers to the inverse of $g_{kl} = g_{kl}(x^i, \rho)$. However, whenever a raised index appears on a coefficient $\overset{m}{h} = \overset{m}{h}(x^i)$, the index is raised w.r.t. $\overset{0}{g}$, i.e. $\overset{m}{h}_j{}^i := \overset{0}{g}{}^{ik} \overset{m}{h}_{kj}$.

Proof. The proof is carried out by induction over m , where we assume $m \leq \frac{n}{2} - 1$ when n is even. When n is odd, we have that $\text{Ric}(\tilde{g}) = (\rho^\infty)$ and when n is even that $\text{Ric}(\tilde{g}) = O(\rho^{\frac{n}{2}-1})$.

Step 1: For $m = 1$, the statement follows from the assumption on \mathbf{P} as well as the contracted version of the second Bianchi identity and $\mathbf{P}^i{}_i = 0$.

Assuming the induction hypothesis that the statement holds for $\overset{b}{h}$ with $1 \leq b \leq m - 1$, we show that the statement also holds for $\overset{m}{h}$. As a preparation, note that as a consequence of the induction hypothesis and parallelity of \mathcal{N} we have

$$(2.11) \quad \overset{u}{h}{}^{ki} \overset{v}{h}_{kj} = 0, \quad \overset{u}{h}{}^{ki} \overset{0}{\nabla}_j \overset{v}{h}_{kl} = 0, \quad \text{for all } 1 \leq u, v \leq m - 1.$$

Moreover, for the inverse g^{ij} of g_{ij} the induction hypothesis implies that

$$(2.12) \quad g^{ij} = \overset{0}{g}{}^{ij} - \sum_{p=1}^{m-1} \frac{1}{p!} \overset{p}{h}{}^{ij} \rho^p + O(\rho^m).$$

Indeed, it is

$$g_{ik} \left(g^{kj} - \sum_{p=1}^{m-1} \frac{1}{p!} h^{kj} \rho^p \right) = \delta_i^j + \sum_{p,q=1}^{m-1} \frac{1}{q!p!} h_{ik} h^{kj} \rho^{p+q} + O(\rho^m),$$

so that the first equation in (2.11) verifies (2.12). Moreover, equations (2.11) and (2.12) then imply that

$$(2.13) \quad \partial_\rho^u (C_{ij}^k)_{\rho=0} = -\frac{1}{2} g^{kl} \overset{0}{\nabla}_i h_{jl}^u = -\frac{1}{2} \overset{0}{\nabla}_i h_j^u, \quad \text{for } i \in \{1, \dots, n-p\} \text{ and } u \leq m-1,$$

where the C_{ij}^k were defined in Section 2.3.

Step 2: Here we show that the induction hypothesis implies that

$$(2.14) \quad \partial_\rho^a R_{ij}|_{\rho=0} = 0, \quad \text{for } a \leq m-1 \text{ and } i \in \{1, \dots, n-p\},$$

where R_{ij} is the Ricci tensor of $g_{ij}(\rho)$. To this end, we rewrite this using (2.8) at $\rho = 0$

$$(2.15) \quad \partial_\rho^a R_{ij} = \partial_\rho^a \left(\overset{0}{\nabla}_i C_{kj}^k - \overset{0}{\nabla}_k C_{ij}^k + C_{ij}^q C_{kq}^k - C_{jk}^q C_{iq}^k \right).$$

Everywhere, not only at $\rho = 0$, we have $C_{kj}^k = -\frac{1}{2} g^{kl} \overset{0}{\nabla}_j g_{kl}$. Expanding the g -s in terms of the $\overset{u}{h}$ using the induction hypothesis as well as (2.11) and (2.12) reveals that $C_{kj}^k = O(\rho^{a+1})$. Thus, the first and third term in (2.15) vanish at $\rho = 0$. The fourth term is treated as follows:

Expanding $\partial_\rho^{m-1} (C_{jk}^q C_{iq}^k)$ at $\rho = 0$ gives a sum of certain coefficients times summands of the form $(\partial_\rho^u C_{jk}^q)(\partial_\rho^v C_{iq}^k)$ with $u+v \leq m-1$. Assuming $i \in \{1, \dots, n-p\}$ and applying (2.13) to this yields

$$(\partial_\rho^u C_{jk}^q)(\partial_\rho^v C_{iq}^k) = \frac{1}{4} g^{pq} g^{kl} \overset{0}{\nabla}_j h_{kp}^u \overset{0}{\nabla}_i h_{ql}^v = 0,$$

since u and v are $\leq m-1$ by the induction hypothesis and the fact that \mathcal{N} is parallel. Thus, the fourth term in (2.15) vanishes at $\rho = 0$.

Finally, we show that the second term in (2.15) vanishes at $\rho = 0$: Assuming $i \in \{1, \dots, n-p\}$ and using (2.13) again, this term is given as

$$(2.16) \quad \frac{1}{2} \overset{0}{\nabla}_k \overset{0}{\nabla}_i h_j^k.$$

By the induction hypothesis, we must necessarily have that $k \in \{1, \dots, p\}$. As \mathcal{N} is $\overset{0}{\nabla}$ -invariant it follows for the curvature of $\overset{0}{g}$

$$(2.17) \quad \overset{0}{R}_{ikhl} = 0 \text{ for all } i \in \{1, \dots, n-p\}, k \in \{1, \dots, p\},$$

see also Lemma 4.1. This shows that the covariant derivatives in (2.16) commute and one obtains $\overset{0}{\nabla}_i$ applied to the divergence of $\overset{a}{h}$, which vanishes by the induction hypothesis. Thus, (2.14) is established.

Now we are going to differentiate the Fefferman-Graham equations (2.3, 2.4, 2.5) with respect to ρ and use that

$$\partial_\rho^k Ric(\tilde{g}) = 0, \quad \text{for all } k \text{ if } n \text{ is odd, and for } k \leq \frac{n}{2} - 2 \text{ if } n \text{ is even.}$$

Step 3: Applying ∂_ρ^{m-2} to the third Fefferman-Graham equation (2.5), where ∂_ρ always denotes the Lie derivative of a tensor in ρ -direction, and then evaluating at $\rho = 0$ yields using (2.11) that

$$(2.18) \quad \overset{0}{g}{}^{klm}{}_{hkl} = 0, \quad \text{for all } m \text{ if } n \text{ is odd and for } m \leq \frac{n}{2} \text{ if } n \text{ is even.}$$

Step 4: We apply ∂_ρ^{m-1} , for $m \leq \frac{n}{2} - 1$ if n is even, to the second Fefferman-Graham equation (2.4) and evaluate at $\rho = 0$. Using (2.18) and rewriting ∇ in terms of $\overset{0}{\nabla}$ and C , the result is

$$(2.19) \quad 0 = \overset{0}{g}{}^{kl}{}_{\overset{0}{\nabla}{}^k h_{il}} + c_{u,v,w} \overset{u}{h}{}^{kl} \left(\partial_\rho^v (C_{ki}^h) \overset{w}{h}_{hl} + \partial_\rho^v (C_{kl}^h) \overset{w}{h}_{ih} - \partial_\rho^v (C_{ik}^h) \overset{w}{h}_{hl} - \partial_\rho^v (C_{il}^h) \overset{w}{h}_{kh} \right)_{\rho=0}$$

for certain integer coefficients $c_{u,v,w}$, where $u + v + w = m$ and $1 \leq w \leq m - 1$. Using $C_{ij}^k = C_{ji}^k$ as well as (2.11), the bracket reduces to

$$(2.20) \quad \overset{u}{h}{}^{kl} \left(\partial_\rho^v (C_{kl}^h) \overset{w}{h}_{ih} \right)_{\rho=0} - \left(\partial_\rho^v (C_{il}^h) \right)_{\rho=0} \overset{w}{h}_h{}^l.$$

In order for the second term in (2.20) to be nonzero, we must necessarily have that $l \in \{1, \dots, p\}$. In this situation, we can insert (2.13) for the C -term and it follows using (2.11) immediately that the resulting term vanishes. It remains to analyze the first term in (2.20). Unwinding the definitions, it is given by

$$(2.21) \quad \overset{u}{h}{}^{kl} \partial_\rho^v \left(g^{hj} \left(\overset{0}{\nabla}_j g_{kl} - \overset{0}{\nabla}_k g_{jl} - \overset{0}{\nabla}_l g_{kj} \right) \right)_{\rho=0} \overset{w}{h}_{ih}.$$

If the ρ -derivative falls on g^{hj} , then the resulting contraction with $\overset{w}{h}_{ih}$ is zero by (2.11). Thus g^{hj} in (2.21) can be replaced by $\overset{0}{g}{}^{hj}$. But then (2.21) involves a factor $\overset{w}{h}_i{}^j$, which can only be nonzero if $j \in \{1, \dots, p\}$, and (2.21) then reduces to

$$(2.22) \quad \overset{u}{h}{}^{kl}{}_{\overset{0}{\nabla}_j h_{kl}} \overset{v}{h}_i{}^j = 0.$$

Thus, every term in (2.19) except for the first one vanishes and we obtain $\overset{0}{\nabla}_k \overset{mk}{h}_i{}^k = 0$, which establishes (2.10).

Step 5: In order to prove (2.9), we apply ∂_ρ^{m-1} to the first Fefferman-Graham equation (2.3), assume that $i \in \{1, \dots, n - p\}$ and evaluate at $\rho = 0$. Using the induction hypothesis and (2.11) applied to the first-fifth term in the Fefferman-Graham equation (2.3), (2.18) applied to the fifth term, as well as (2.14), we obtain that at $\rho = 0$ and for $i \in \{1, \dots, n - p\}$ that

$$(2.23) \quad \left(m - \frac{n}{2} \right) \overset{m}{h}_{ij} + \partial_\rho^{m-1} Ric_{ij}|_{\rho=0} = \left(m - \frac{n}{2} \right) \overset{m}{h}_{ij} = \partial_\rho^{m-1} (Ric_{ij}(\tilde{g}))|_{\rho=0}.$$

If n is odd, $\partial_\rho^{m-1}(\text{Ric}_{ij}(\tilde{g})|_{\rho=0}) = 0$ for all m and hence equation (2.23) shows that ${}^m h_{ij} = 0$ for $i = 1, \dots, n-p$ completing the induction and establishing (2.9) for all m .

If n is even, $\partial_\rho^{m-1}(\text{Ric}_{ij}(\tilde{g})|_{\rho=0}) = 0$ for all $m \leq \frac{n}{2} - 1$, and hence equation (2.23) shows that ${}^m h_{ij} = 0$ for $i = 1, \dots, n-p$ for all $m \leq \frac{n}{2} - 1$. But by taking $m = \frac{n}{2}$, it also gives a formula for the obstruction tensor \mathcal{O}_{ij} , in which c_n is a non-zero constant:

$$\mathcal{O}_{ij} = c_n \partial_\rho^{\frac{n}{2}-1} \widetilde{\text{Ric}}_{ij}|_{\rho=0} = c_n \partial_\rho^{\frac{n}{2}-1} R_{ij}|_{\rho=0} = 0,$$

if $i \in \{1, \dots, n-p\}$ by (2.14). This verifies the statement about the obstruction tensor.

Finally, in the case that n is even, the terms ${}^m h_{ij}$, for $m \geq \frac{n}{2}$, in an ambient metric are not subject to any equation and we can choose them to be divergence free and with image in \mathcal{N} . This completes the proof of the Theorem. \square

3. TOWARDS LINEAR FEFFERMAN-GRAHAM EQUATIONS

In this section we will compute the Ricci tensor for metrics of the form

$$\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2(\overset{0}{\mathbf{g}} + \mathbf{h}),$$

where $\mathbf{h} = \mathbf{h}(\rho)$ is a ρ -dependent family of symmetric bilinear forms with $\mathbf{h}|_{\rho=0} = 0$ and moreover with the property that

$$\text{Im}(\mathbf{h}) \subset \mathcal{N},$$

for a totally null distribution \mathcal{N} . This is motivated by our aim to find the ambient metrics for metrics with two-step nilpotent Schouten tensor \mathbf{P} . As we have seen in Lemma 2.1, if the Schouten tensor \mathbf{P} is two-step nilpotent, its image is contained in a totally null vector distribution \mathcal{N} . On the other hand from [5] we know that $\dot{\mathbf{h}}|_{\rho=0} = 2\mathbf{P}$, which leads to our ansatz $\text{Im}(\mathbf{h}(\rho)) \subset \mathcal{N}$ for all ρ . We will then successively impose further conditions on \mathcal{N} and on \mathbf{h} so that the Fefferman-Graham equations become at most quadratic and eventually linear in \mathbf{h} .

A *remark on notation* is in place: here and in the following we will use the boldface letters $\overset{0}{\mathbf{g}}, \mathbf{g}, \mathbf{h}$ for tensors, because it allows us to distinguish between the tensor $\overset{0}{\mathbf{g}}, \mathbf{g}, \mathbf{h}$ and its components $g_{ij}, \overset{0}{g}_{ij}, h_{ij}$ in a *specific* basis that is adapted to \mathcal{N} (see the next subsection) and later on satisfies additional properties. Some of the statements in the next sections will only hold for the components h_{ij} of \mathbf{h} in such a basis.

3.1. Conventions. Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathcal{N} be a vector distribution that is totally null and of rank $p > 1$. We fix a local frame

$$(3.1) \quad \mathbf{e}_1, \dots, \mathbf{e}_n \quad \text{such that } \mathbf{e}_1, \dots, \mathbf{e}_p \text{ span } \mathcal{N} \text{ and } \mathbf{e}_1, \dots, \mathbf{e}_{n-p} \text{ span } \mathcal{K} = \mathcal{N}^\perp.$$

We will use the following index conventions:

$$(3.2) \quad \begin{aligned} i, j, k, \dots &\in \{1, \dots, n\} \\ a, b, c, \dots &\in \{1, \dots, p\} \\ A, B, C, \dots &\in \{p+1, \dots, n-p\} \\ \bar{a}, \bar{b}, \bar{c}, \dots &\in \{n-p+1, \dots, n\}. \end{aligned}$$

In the following we will chose the frame such that

$$(3.3) \quad \begin{aligned} \overset{0}{\mathbf{g}}(\mathbf{e}_{\bar{a}}, \mathbf{e}_b) &= \overset{0}{g}(\mathbf{e}_b, \mathbf{e}_{\bar{a}}) = \overset{0}{g}_{\bar{a}b} = \overset{0}{g}_{b\bar{a}} \text{ constant and non degenerate,} \\ \overset{0}{\mathbf{g}}(\mathbf{e}_A, \mathbf{e}_B) &= \overset{0}{g}(\mathbf{e}_B, \mathbf{e}_A) = \overset{0}{g}_{AB} = \overset{0}{g}_{BA} \text{ constant and non degenerate,} \\ \overset{0}{\mathbf{g}}(\mathbf{e}_i, \mathbf{e}_j) &= 0 \text{ otherwise.} \end{aligned}$$

In other words, if $\Theta^1, \dots, \Theta^n$ denote the algebraic duals to the \mathbf{e}_i 's, i.e.

$$\Theta^i(\mathbf{e}_j) = \delta^i_j$$

then the metric is

$$(3.4) \quad \overset{0}{\mathbf{g}} = \overset{0}{g}_{ij} \Theta^i \Theta^j = 2 \overset{0}{g}_{a\bar{c}} \Theta^a \Theta^{\bar{c}} + \overset{0}{g}_{AB} \Theta^A \Theta^B.$$

Note that the inverse g^{ij} of the matrix $\overset{0}{g}_{ij}$ is given by $g^{a\bar{b}} = g^{\bar{b}a}$ and g^{AB} satisfying

$$\overset{0}{g}_{a\bar{b}} g^{\bar{b}c} = \delta_a^c, \quad \overset{0}{g}_{\bar{a}b} g^{b\bar{c}} = \delta_{\bar{a}}^{\bar{c}}, \quad \overset{0}{g}_{AB} g^{BC} = \delta_A^C.$$

This relates the algebraic duals Θ^i to the metric duals $\overset{0}{\mathbf{g}}(\mathbf{e}_i, \cdot)$ of \mathbf{e}_i as follows

$$\Theta^a = \overset{0}{g}^{a\bar{c}} \overset{0}{\mathbf{g}}(\mathbf{e}_{\bar{c}}, \cdot), \quad \Theta^{\bar{a}} = \overset{0}{g}^{\bar{a}c} \overset{0}{\mathbf{g}}(\mathbf{e}_c, \cdot), \quad \Theta^A = \overset{0}{g}^{AB} \overset{0}{\mathbf{g}}(\mathbf{e}_B, \cdot)$$

Now we consider a symmetric bilinear form \mathbf{h} (possibly depending on a parameter ρ) that satisfies

$$\text{Im}(\mathbf{h}^\sharp) \subset \mathcal{N},$$

where \mathbf{h}^\sharp denotes the metric dual to \mathbf{h} , $\mathbf{h}(X, Y) = \overset{0}{\mathbf{g}}(\mathbf{h}^\sharp(X), Y)$. This is equivalent to \mathbf{h} being of the form

$$(3.5) \quad \mathbf{h} := h_{\bar{a}\bar{c}} \Theta^{\bar{a}} \Theta^{\bar{c}} = h_{ij} \Theta^i \circ \Theta^j,$$

i.e., $h_{ij} = 0$ unless $i, j = \bar{a}, \bar{c}$, for smooth functions $h_{\bar{a}\bar{c}} = h_{\bar{a}\bar{c}}(\rho, x)$ with $h_{\bar{a}\bar{c}} = h_{\bar{c}\bar{a}}$. The corresponding $(1, 1)$ tensor \mathbf{h}^\sharp has components

$$h_{\bar{a}}^b = g^{b\bar{c}} h_{\bar{a}\bar{c}}$$

and all others zero, i.e.

$$\mathbf{h}^\sharp = h_{\bar{a}}^b \Theta^{\bar{a}} \otimes \mathbf{e}_b.$$

and satisfies

$$(\mathbf{h}^\sharp)^2 = 0, \quad \text{i.e. } h_{\bar{a}}^k h_k^b = 0.$$

It holds that

$$\mathcal{K} = \mathcal{N}^\perp \subset \ker(\mathbf{h}^\sharp).$$

Finally, we obtain the $(2, 0)$ -tensor defined by $h^{ij} = g^{ik} g^{jl} h_{kl}$, i.e., with

$$h^{bd} = g^{b\bar{a}} g^{d\bar{c}} h_{\bar{a}\bar{c}}$$

and all other components zero. From now on the components of all the tensor are given in the frame (3.1) with the index conventions as in (3.3). We have

Lemma 3.1. For \mathbf{h} as in (3.5) denote by $\mathbf{h}^{(r)} = (h_{ij}^{(r)})$ the tensor whose components are given by the r -th ∂_ρ -derivative of the components of h_{ij} , i.e., $\mathbf{h}^{(r)} := \partial_\rho^r(h_{ij})\Theta^i \circ \Theta^j$. Then

$$(3.6) \quad {}^0g^{ij}h_{ij}^{(r)} = 0 \quad \text{and} \quad h_{ik}^{(r)}h_j^{(s)k} = 0 \quad \text{for all } 0 \leq r, s.$$

Moreover, if ${}^0\nabla$ is the Levi-Civita connection of ${}^0\mathbf{g}$, then

$$(3.7) \quad {}^0\nabla_k h_{ij}^{(r)} = 0, \quad \text{unless } i = \bar{a} \text{ or } j = \bar{a},$$

as well as

$$(3.8) \quad {}^0g^{kl}{}^0\nabla_i h_{kl}^{(r)} = 0,$$

and

$$(3.9) \quad h_i^{(r)l} \nabla_k h_{jl}^{(s)} = -h_j^{(s)l} \nabla_k h_{il}^{(r)}$$

for all $r, s = 0, 1, \dots$

Proof. Equations (3.6) follows from the fact that h_i^j squares to zero and is trace free. Equation (4.9) follows from

$${}^0\nabla_X \mathbf{h}(\mathbf{e}_i, \mathbf{e}_j) = X(\mathbf{h}(\mathbf{e}_i, \mathbf{e}_j)) - \mathbf{h}({}^0\nabla_X \mathbf{e}_i, \mathbf{e}_j) - \mathbf{h}(\mathbf{e}_i, {}^0\nabla_X \mathbf{e}_j) = 0$$

unless \mathbf{e}_i or \mathbf{e}_j is equal to $\mathbf{e}_{\bar{a}}$.

The last equation (3.9) follows from (3.6),

$$0 = \nabla_k \left(h_i^{(r)l} h_{jl}^{(s)} \right) = h_i^{(r)l} \nabla_k h_{jl}^{(s)} + h_j^{(s)l} \nabla_k h_{il}^{(r)}.$$

by the Leibniz rule. \square

3.2. The Ricci tensor of a 2-step nilpotent perturbation. In the following, for a semi-Riemannian metric ${}^0\mathbf{g}$ we will consider perturbations by a 2-step nilpotent, symmetric bilinear form \mathbf{h} depending on a parameter ρ . By the results in the previous section we can write this perturbation as

$$(3.10) \quad \mathbf{g} = {}^0\mathbf{g} + \mathbf{h}, \quad \text{where} \quad \mathbf{h} = h_{\bar{a}\bar{c}}\Theta^{\bar{a}} \circ \Theta^{\bar{c}} \quad \text{and} \quad {}^0\mathbf{g} = {}^0g_{\bar{a}\bar{b}}\Theta^{\bar{a}}\Theta^{\bar{b}} + {}^0g_{AB}\Theta^A\Theta^B$$

where we use the conventions in Section 3.1 and with smooth functions $h_{\bar{a}\bar{c}} = h_{\bar{a}\bar{c}}(\rho, x)$ with $h_{\bar{a}\bar{c}} = h_{\bar{c}\bar{a}}$. The metric coefficients of \mathbf{g} are $g_{ij}(\rho, x) := {}^0g_{ij}(x) + h_{ij}(\rho, x)$. The perturbed metric \mathbf{g} has the property that the inverse of \mathbf{g} is linear in the perturbation \mathbf{h} , i.e., if g^{ij} are the coefficient of the inverse of g_{ij} then

$$(3.11) \quad g^{ij} = {}^0g^{ij} - h^{ij}.$$

In the following we will raise the indices with ${}^0g_{ij}$. First we observe:

Proposition 3.1. Let ${}^0\mathbf{g}$ be a semi-Riemannian metric and \mathbf{h} a ρ -dependent, 2-step nilpotent symmetric bilinear form. Then for the metric

$$(3.12) \quad \tilde{\mathbf{g}} = 2d(\rho t)dt + t^2({}^0\mathbf{g} + \mathbf{h})$$

the possibly non-vanishing components of the Ricci tensor are given by

$$(3.13) \quad g^{kl} \overset{0}{\nabla}_k \dot{h}_{il} \quad \text{and} \quad \rho \ddot{h}_{ij} - \left(\frac{n}{2} - 1\right) \dot{h}_{ij} + R_{ij}.$$

Here the dots denote the ρ derivatives of the h_{ij} 's and R_{ij} are the components of the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$.

Proof. The components of the Ricci tensor of $\widetilde{\mathbf{g}}$ are given by the left-hand sides of the Fefferman-Graham equations (2.3, 2.4, 2.5). Lemma 3.1 shows that the term in the third Fefferman-Graham equation (2.5) is zero.

In order to analyse the term in the second Fefferman-Graham equation (2.4), we use formula (2.6) for expressing ∇ in terms of $\overset{0}{\nabla}$ and the tensor $C_{ij}^k = C_{ji}^k$, i.e.,

$$(3.14) \quad \begin{aligned} g^{kl} (\nabla_k \dot{g}_{il} - \nabla_i \dot{g}_{kl}) &= (g^{kl} - h^{kl}) (\overset{0}{\nabla}_k \dot{h}_{il} - \overset{0}{\nabla}_i \dot{h}_{kl} + C_{kl}^p \dot{h}_{ip} - C_{il}^p \dot{h}_{pk}) \\ &= g^{kl} (\overset{0}{\nabla}_k \dot{h}_{il} + C_{kl}^p \dot{h}_{ip} - C_{il}^p \dot{h}_{pk}) - h^{kl} \overset{0}{\nabla}_k \dot{h}_{il} - h^{kl} C_{kl}^p \dot{h}_{ip} \end{aligned}$$

because \mathbf{h} is trace free and because of Lemma 3.1. For C_{ij}^k , the formula (2.7) reduces to

$$(3.15) \quad C_{ij}^k = \frac{1}{2} (g^{kl} - h^{kl}) (\overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_i h_{jl} - \overset{0}{\nabla}_j h_{il})$$

again by Lemma 3.1. Hence

$$\dot{h}_{kp} C_{ij}^p = \frac{1}{2} \dot{h}_k^l (\overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_i h_{jl} - \overset{0}{\nabla}_j h_{il})$$

Therefore the last term in (3.14) becomes

$$\begin{aligned} 2h^{kl} \dot{h}_{ip} C_{kl}^p &= h^{kl} \dot{h}_i^p (\nabla_p h_{kl} - \nabla_k h_{pl} - \nabla_l h_{pk}) \\ &= -h^{kl} (h_{kl} \nabla_p \dot{h}_i^p - h_{pl} \nabla_k \dot{h}_i^p - h_{pk} \nabla_l \dot{h}_i^p) = 0 \end{aligned}$$

because of (3.9) in Lemma 3.1. Similarly, the remaining term in (3.14) is

$$g^{kl} (C_{kl}^p \dot{h}_{ip} - C_{il}^p \dot{h}_{pk}) - h^{kl} \overset{0}{\nabla}_k \dot{h}_{il} = -\dot{h}_i^l \overset{0}{\nabla}_k h_l^k - h^{kl} \overset{0}{\nabla}_k \dot{h}_{il} + \frac{1}{2} \dot{h}_{kl} \nabla_i h^{kl} = 0.$$

This verifies the formula for the terms in the second Fefferman-Graham equation. The term in the first Fefferman-Graham equation (2.3) is seen to be equal to the second term in (3.13) by using Lemma 3.1. \square

The lemma shows that, apart from the Ricci tensor of \mathbf{g} , the Fefferman-Graham equations contain only terms that are linear in \mathbf{h} . Thus, we now determine the Ricci tensor of a metric $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ in terms of the Ricci tensor of $\overset{0}{\mathbf{g}}$ and of \mathbf{h} using formula (2.8) and apply this to a metric $\widetilde{\mathbf{g}} = 2d(\rho t)dt + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$. For this we note that for a metric as in (3.10) with inverse (3.11) the formula (2.8) for the Ricci tensor of \mathbf{g} contains terms up to fourth order in \mathbf{h} . Hence we observe:

Proposition 3.2. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathbf{h} be a 2-step nilpotent symmetric bilinear form. The Ricci tensor R_{ij} of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ is given by*

$$(3.16) \quad R_{ij} = \overset{0}{R}_{ij} + \overset{0}{\nabla}^k \overset{0}{\nabla}_{(i} h_{j)k} - \frac{1}{2} \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij} + Q_{ij}^{(2)}(\mathbf{h}) + Q_{ij}^{(3)}(\mathbf{h}) + Q_{ij}^{(4)}(\mathbf{h})$$

in which we raise the indices with $\overset{0}{g}_{ij}$ and where the $Q_{ij}^{(r)}(\mathbf{h})$ are symmetric tensors that are of order $r = 2, 3, 4$ in h_{ij} , and which are given explicitly in (3.20), (3.19) and (3.18) below.

Now we are going to compute the $Q_{ij}^{(k)}(\mathbf{h})$'s by using equation (2.8) for the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$. First we note that the formula (3.15) for C_{ij}^k and Lemma 3.1 implies

$$C_{ki}^k = -\frac{1}{2}(\overset{0}{g}^{kl} - h^{kl})\overset{0}{\nabla}_i h_{kl} = 0.$$

Hence (2.8) simplifies to

$$(3.17) \quad R_{ij} = \overset{0}{R}_{ij} - \overset{0}{\nabla}_k C_{ij}^k - C_{jk}^p C_{ip}^k.$$

We start with the terms of fourth order in \mathbf{h} : by (3.9) in Lemma 3.1 we get

$$(3.18) \quad \begin{aligned} Q_{ij}^{(4)}(\mathbf{h}) &= -\frac{1}{4}h^{pq}h^{kl}(\overset{0}{\nabla}_q h_{jk} - \overset{0}{\nabla}_j h_{kq} - \overset{0}{\nabla}_k h_{jq})(\overset{0}{\nabla}_l h_{ip} - \overset{0}{\nabla}_i h_{lp} - \overset{0}{\nabla}_p h_{il}) \\ &= -\frac{1}{4}h^{pq}h^{kl}(\overset{0}{\nabla}_q h_{jk} - \overset{0}{\nabla}_k h_{jq})(\overset{0}{\nabla}_l h_{ip} - \overset{0}{\nabla}_p h_{il}) \\ &= \frac{1}{4}h^{ab}h^{cd}(\overset{0}{\nabla}_c h_{jb} - \overset{0}{\nabla}_b h_{jc})(\overset{0}{\nabla}_d h_{ia} - \overset{0}{\nabla}_a h_{id}) \\ &= \frac{1}{4}h^{ab}h^{cd}(\mathbf{h}(\mathbf{e}_j, [\mathbf{e}_c, \mathbf{e}_b]))(\mathbf{h}(\mathbf{e}_i, [\mathbf{e}_d, \mathbf{e}_a])), \end{aligned}$$

where, for the last equality, we have written the summation in terms of the frame field \mathbf{e}_i and used that $h_{ia} = 0$. Note that $Q_{ij}^{(4)}(\mathbf{h}) = 0$ if $[\mathbf{e}_a, \mathbf{e}_b] \in \mathcal{K}$.

Now we compute the third order terms and because of (4.9) in Lemma 3.1 we obtain

$$(3.19) \quad \begin{aligned} Q_{ij}^{(3)}(\mathbf{h}) &= -\frac{1}{2}h^{kl}\overset{0}{g}^{pq} \left(\overset{0}{\nabla}_i h_{kp} \overset{0}{\nabla}_j h_{lq} - (\overset{0}{\nabla}_p h_{ik} - \overset{0}{\nabla}_k h_{ip})(\overset{0}{\nabla}_q h_{jl} - \overset{0}{\nabla}_l h_{jq}) \right) \\ &= -\frac{1}{2}h^{ab}\overset{0}{g}^{\bar{c}d} \left(\overset{0}{\nabla}_i h_{a\bar{c}} \overset{0}{\nabla}_j h_{bd} - (\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}})(\overset{0}{\nabla}_d h_{jb} - \overset{0}{\nabla}_b h_{jd}) \right) \\ &\quad -\frac{1}{2}h^{ab}\overset{0}{g}^{CD} \left(\overset{0}{\nabla}_i h_{aC} \overset{0}{\nabla}_j h_{bD} - (\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})(\overset{0}{\nabla}_D h_{jb} - \overset{0}{\nabla}_b h_{jD}) \right) \\ &= \frac{1}{2}h^{ab}\overset{0}{g}^{\bar{c}d}(\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}})(\overset{0}{\nabla}_d h_{jb} - \overset{0}{\nabla}_b h_{jd}) \\ &\quad + \frac{1}{2}h^{ab}\overset{0}{g}^{CD} \left((\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})(\overset{0}{\nabla}_D h_{jb} - \overset{0}{\nabla}_b h_{jD}) \right) \\ &= \frac{1}{2}h^{ab} \left(\overset{0}{g}^{\bar{c}d}(\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}})\mathbf{h}(\mathbf{e}_j, [\mathbf{e}_d, \mathbf{e}_b]) \right) \\ &\quad + \frac{1}{2}h^{ab} \left(\overset{0}{g}^{CD}((\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})\mathbf{h}(\mathbf{e}_j, [\mathbf{e}_b, \mathbf{e}_D])) \right). \end{aligned}$$

Clearly, this vanishes if $[\mathbf{e}_a, \mathbf{e}_b] \in \mathcal{K}$ and $[\mathbf{e}_a, \mathbf{e}_B] \in \mathcal{K}$, and in particular if \mathcal{K} is involutive.

Finally, we turn to the second order terms. They are given as

$$(3.20) \quad \begin{aligned} Q_{ij}^{(2)}(\mathbf{h}) &= \overset{0}{\nabla}_k h^{kl} \left(\frac{1}{2}\overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_{(i} h_{j)l} \right) + h^{kl} \left(\frac{1}{2}\overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_k \overset{0}{\nabla}_{(i} h_{j)l} \right) \\ &\quad - \frac{1}{4}\overset{0}{\nabla}_i h^{kl} \overset{0}{\nabla}_j h_{kl} - \frac{1}{4} \left(\overset{0}{\nabla}^k h_i{}^l - \overset{0}{\nabla}^l h_i{}^k \right) \left(\overset{0}{\nabla}_l h_{jk} - \overset{0}{\nabla}_k h_{jl} \right). \end{aligned}$$

First we rewrite the last term as

$$\frac{1}{4} \left(\overset{0}{\nabla}^k h_i{}^l - \overset{0}{\nabla}^l h_i{}^k \right) \left(\overset{0}{\nabla}_l h_{jk} - \overset{0}{\nabla}_k h_{jl} \right) = \overset{0}{\nabla}_{[k} h_{l]i} \overset{0}{\nabla}^k h_j{}^l = \overset{0}{\nabla}_{[k} h_{l]j} \overset{0}{\nabla}^k h_i{}^l.$$

Next, we analyse the term $h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_{(i}h_{j)l}$ using the divergence of \mathbf{h} , Lemma 3.1, the curvature and the fact that \mathbf{h} is 2-step nilpotent:

$$\begin{aligned} h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_ih_{jl} &= -h_{jl}\overset{0}{\nabla}_k\overset{0}{\nabla}_ih^{kl} - \overset{0}{\nabla}_kh_{lj}\overset{0}{\nabla}_ih^{kl} - \overset{0}{\nabla}_kh^{kl}\overset{0}{\nabla}_ih_{jl} \\ &= -h_{jl}\left(\overset{0}{\nabla}_i\overset{0}{\nabla}_kh^{kl} + h^{pl}\overset{0}{R}_{ki}{}^k{}_p + h^{kp}\overset{0}{R}_{ki}{}^l{}_p\right) - \overset{0}{\nabla}_kh_{lj}\overset{0}{\nabla}_ih^{kl} - \overset{0}{\nabla}_kh^{kl}\overset{0}{\nabla}_ih_{jl} \\ &= -h_{jl}\overset{0}{\nabla}_i\overset{0}{\nabla}_kh^{kl} - h_j{}^lh^{kp}\overset{0}{R}_{kilp} - \overset{0}{\nabla}_kh_{lj}\overset{0}{\nabla}_ih^{kl} - \overset{0}{\nabla}_kh^{kl}\overset{0}{\nabla}_ih_{jl} \end{aligned}$$

Hence, we obtain

$$\begin{aligned} Q_{ij}^{(2)}(\mathbf{h}) &= \frac{1}{2}\overset{0}{\nabla}_kh^{kl}\overset{0}{\nabla}_lh_{ij} + h_{l(i}\overset{0}{\nabla}_{j)}\overset{0}{\nabla}_kh^{kl} + \frac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_lh_{ij} - h^{kp}h^l{}_{(i}\overset{0}{R}_{j)klp} + \overset{0}{\nabla}_kh_{l(i}\overset{0}{\nabla}_{j)}h^{kl} \\ &\quad - \frac{1}{4}\overset{0}{\nabla}_ih^{kl}\overset{0}{\nabla}_jh_{kl} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}{}^kh_j{}^l \end{aligned}$$

Therefore, if \mathbf{h} is divergence free, i.e. $\overset{0}{\nabla}_kh^{kl} = 0$, we get formula (3.21) for $Q_{ij}^{(2)}(\mathbf{h})$.

Proposition 3.3. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathbf{h} be a 2-step nilpotent symmetric bilinear form such that there is a totally null distribution \mathcal{N} with $\text{Im}(\mathbf{h}) \subset \mathcal{N}$ and $\mathcal{K} = \mathcal{N}^\perp$ involutive. Then the Ricci tensor R_{ij} of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ is at most quadratic in \mathbf{h} , i.e., the terms $Q_{ij}^{(3)}(\mathbf{h})$ and $Q_{ij}^{(4)}(\mathbf{h})$ in (3.16) vanish. If we assume in addition that \mathbf{h} is divergence free, then*

$$(3.21) \quad Q_{ij}^{(2)}(\mathbf{h}) = \frac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_lh_{ij} - h^{kp}h^l{}_{(i}\overset{0}{R}_{j)klp} + \overset{0}{\nabla}_kh_{l(i}\overset{0}{\nabla}_{j)}h^{kl} - \frac{1}{4}\overset{0}{\nabla}_ih^{kl}\overset{0}{\nabla}_jh_{kl} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}{}^kh_j{}^l.$$

Now we are looking for geometric conditions such that $Q_{ij}^{(2)}(\mathbf{h})$ simplifies further and perhaps vanishes. In fact we show:

Theorem 3.1. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathbf{h} be a divergence free, 2-step nilpotent symmetric bilinear form. If there is an involutive distribution \mathcal{K} with $\text{Im}(\mathbf{h}) \subset \mathcal{K}^\perp \subset \mathcal{K}$ and*

$$(3.22) \quad \overset{0}{\nabla}_ZY \in \mathcal{K}^\perp, \quad \text{for all } Y, Z \in \mathcal{K}^\perp$$

$$(3.23) \quad \overset{0}{\nabla}_XY \in \mathcal{K}, \quad \text{for all } X \in TM, Y \in \mathcal{K}^\perp,$$

then,

$$(3.24) \quad Q_{ij}^{(2)}(\mathbf{h}) = \frac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_lh_{ij} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}{}^kh_j{}^l.$$

Moreover, if in addition

$$(3.25) \quad \mathcal{L}_Y\mathbf{h} = 0, \quad \text{for all } Y \in \mathcal{K}^\perp,$$

then $Q_{ij}^{(2)}$ is zero, i.e., the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ is linear in the perturbation \mathbf{h} ,

$$(3.26) \quad R_{ij} = \overset{0}{R}_{ij} + \overset{0}{\nabla}{}^k\overset{0}{\nabla}_{(i}h_{j)k} - \frac{1}{2}\overset{0}{\nabla}{}^k\overset{0}{\nabla}_kh_{ij}.$$

Proof. We work in a basis $(\mathbf{e}_a, \mathbf{e}_A, \mathbf{e}_{\bar{a}})$ and use the conventions as in Section 3.1. First note that assumption (3.23) implies that terms of the form $\overset{0}{\nabla}_k h_{al}$ or $\overset{0}{\nabla}_k h_{AB}$ are zero (where we use our index convention). This implies that in formula (3.21) for $Q_{ij}^{(2)}(\mathbf{h})$ the terms $\overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}_j h^{kl}$ and $\overset{0}{\nabla}_i h^{kl} \overset{0}{\nabla}_j h_{kl}$ vanish.

Next we look at the curvature term in formula (3.21) for $Q_{ij}^{(2)}(\mathbf{h})$. Again by assumption (3.23) we have

$$\overset{0}{R}(\mathbf{e}_i, \mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c) = -\overset{0}{g}(\overset{0}{\nabla}_{\mathbf{e}_a} \mathbf{e}_b, \overset{0}{\nabla}_{\mathbf{e}_i} \mathbf{e}_c) + \overset{0}{g}(\overset{0}{\nabla}_{\mathbf{e}_i} \mathbf{e}_b, \overset{0}{\nabla}_{\mathbf{e}_a} \mathbf{e}_c)$$

which vanishes because of (3.22) and (3.23). This proves the first statement.

To prove the second point, assumption (3.23) gives

$$(3.27) \quad \overset{0}{\nabla}_{[k} h_{l]i} \overset{0}{\nabla}^k h_j{}^l = -\frac{1}{2} \overset{0}{g}{}^{\bar{a}b} \overset{0}{g}{}^{\bar{c}d} \overset{0}{\nabla}_d h_{\bar{a}i} \overset{0}{\nabla}_b h_{\bar{c}j} + \frac{1}{2} \overset{0}{g}{}^{AB} \overset{0}{g}{}^{CD} (\mathbf{h}([e_A, e_C], \mathbf{e}_i) \overset{0}{\nabla}_B h_{Dj}.$$

Note that the last term in this formula is zero since \mathcal{K} is involutive. On the other hand, we observe that for $Y \in \mathcal{K}^\perp$

$$\overset{0}{\nabla}_Y \mathbf{h} = \mathcal{L}_Y \mathbf{h},$$

because of (3.23). This also shows that in our situation $\mathcal{L}_Y \mathbf{h}$ is tensorial in $Y \in \mathcal{K}^\perp$. If we now assume that $\mathcal{L}_Y \mathbf{h} = 0$ for all $Y \in \mathcal{K}^\perp$, then $\overset{0}{\nabla}_Y \mathbf{h} = 0$ for all $Y \in \mathcal{K}^\perp$ and thus the remaining term in (3.27) vanishes, as well as the term $h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij}$. Consequently, $Q_{ij}^{(2)}(\mathbf{h})$ is zero. \square

We can apply these results to the metric $\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2 \mathbf{g}$ as defined in (3.12): Under the assumption that \mathcal{K} is involutive and that \mathbf{h} is divergence free we can apply Proposition 3.1. Since $\dot{\mathbf{h}}$ is divergence free if \mathbf{h} is divergence free, it implies that $\tilde{\mathbf{g}}$ is Ricci flat if and only if

$$(3.28) \quad \rho \ddot{h}_{ij} - \left(\frac{n}{2} - 1\right) \dot{h}_{ij} + \overset{0}{\nabla}^k \overset{0}{\nabla}_{(i} h_{j)k} - \frac{1}{2} \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij} + \overset{0}{R}_{ij} + Q_{ij}^{(2)}(\mathbf{h}) = 0,$$

where $Q_{ij}^{(2)}(\mathbf{h})$ is given as in (3.21). Moreover, that \mathbf{h} is divergence free also allows us to simplify the term $\overset{0}{\nabla}^k \overset{0}{\nabla}_{(i} h_{j)k}$. In fact, if $\overset{0}{\nabla}^k h_{ik} = 0$ we get

$$(3.29) \quad \overset{0}{\nabla}^k \overset{0}{\nabla}_i h_{jk} = \overset{0}{R}{}^k{}_{ij} h_{kl} + \overset{0}{R}{}^k{}_{ik} h_{jl} + \overset{0}{\nabla}_i \overset{0}{\nabla}^k h_{jk} = \overset{0}{R}{}^k{}_{ij} h_{kl} + \overset{0}{R}_i{}^l h_{jl}.$$

This shows that we can eliminate all $\overset{0}{\nabla}_i$ derivatives from this term to obtain

Corollary 3.1. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathbf{h} be a 2-step nilpotent symmetric bilinear form such that there is an involutive distribution \mathcal{K} such that $\text{Im}(\mathbf{h}) \subset \mathcal{K}^\perp \subset \mathcal{K}$. Then the metric $\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$ is Ricci-flat if the perturbation \mathbf{h} is divergence free and*

$$(3.30) \quad \rho \ddot{h}_{ij} - \left(\frac{n}{2} - 1\right) \dot{h}_{ij} - \frac{1}{2} \overset{0}{\square} h_{ij} + \overset{0}{R}{}^k{}_{ij} h_{kl} + \overset{0}{R}{}^k{}_{(i} h_{j)k} + \overset{0}{R}_{ij} + Q_{ij}^{(2)}(\mathbf{h}) = 0,$$

where $Q_{ij}^{(2)}(\mathbf{h})$ is given in (3.21) and $\overset{0}{\square} h_{ij} = \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij}$.

Theorem 3.1 gives another

Corollary 3.2. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and \mathbf{h} be a 2-step nilpotent symmetric bilinear form. If there is a totally null distribution \mathcal{N} such that $\text{Im}(\mathbf{h}) \subset \mathcal{N}$, $\mathcal{K} = \mathcal{N}^\perp$ is involutive and conditions (3.22) and (3.23) of Theorem 3.1 are satisfied, then the metric $\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$ is Ricci-flat if the following system of linear PDEs on $\mathbf{h} = (h_{ij})$ is satisfied:*

$$(3.31) \quad \text{div}(\mathbf{h}) = 0,$$

$$(3.32) \quad \mathcal{L}_Y \mathbf{h} = 0, \quad \forall Y \in \mathcal{K}^\perp,$$

$$(3.33) \quad \rho \ddot{h}_{ij} - \left(\frac{n}{2} - 1\right) \dot{h}_{ij} - \frac{1}{2} \overset{0}{\square} h_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}{}^k{}_{(i} h_{j)k} + \overset{0}{R}{}_{ij} = 0.$$

Remark 3.1. In this remark we will explain how the assumptions in Theorem 3.1 can be expressed in terms of Lie brackets for a frame $\mathbf{e}_a, \mathbf{e}_B, \mathbf{e}_{\bar{c}}$. First we show that if, in addition to the integrability of \mathcal{K} , we assume the bracket relations

$$(3.34) \quad [\mathbf{e}_a, \mathbf{e}_{\bar{c}}] \in \mathcal{K}$$

$$(3.35) \quad [\mathbf{e}_a, \mathbf{e}_b] = 0$$

then the assumptions (3.22) and (3.23) in Theorem 3.1 are satisfied:

Lemma 3.2. *Let $\overset{0}{\mathbf{g}}$ be a metric in the form (3.4). Assume that \mathcal{K} is involutive and that $[\mathbf{e}_a, \mathbf{e}_{\bar{c}}] \in \mathcal{K}$.*

- (i) *Then $[\mathbf{e}_a, \mathbf{e}_b] = 0$ for all $a, b = 1, \dots, p$ if and only if $\overset{0}{\nabla}_X \mathbf{e}_a \in \mathcal{K}$ for all $X \in TM$.*
- (ii) *If one of the conditions in (i) is satisfied, then*

$$\overset{0}{\nabla}_a \mathbf{e}_b = 0, \quad \text{for all } a, b = 1, \dots, p,$$

or equivalently

$$\overset{0}{\nabla}_a \Theta^{\bar{c}} = 0, \quad \text{for all } a = 1, \dots, p, \bar{c} = n - p + 1, \dots, n.$$

Proof. Under the given two conditions, the Koszul formula gives immediately that

$$2\overset{0}{\mathbf{g}}(\overset{0}{\nabla}_i \mathbf{e}_a, \mathbf{e}_b) = \overset{0}{\mathbf{g}}([\mathbf{e}_i, \mathbf{e}_a], \mathbf{e}_b) + \overset{0}{\mathbf{g}}([\mathbf{e}_i, \mathbf{e}_b], \mathbf{e}_a) + \overset{0}{\mathbf{g}}([\mathbf{e}_a, \mathbf{e}_b], \mathbf{e}_i) = \overset{0}{\mathbf{g}}([\mathbf{e}_a, \mathbf{e}_b], \mathbf{e}_i)$$

for all $i, j = 1, \dots, p$ and $i = 1, \dots, n$. This shows that $\overset{0}{\nabla}_{\mathbf{e}_{\bar{c}}} \mathbf{e}_a \in \mathcal{K}$ if and only if $[\mathbf{e}_a, \mathbf{e}_b] = 0$. This computation also shows the second statement. \square

Moreover, note that when assuming the bracket relations (3.34) and (3.35), the Lie derivative simplifies to

$$\mathcal{L}_{\mathbf{e}_{\bar{a}}} \mathbf{h}(\mathbf{e}_i, \mathbf{e}_j) = dh_{ij}(\mathbf{e}_a).$$

Hence, under the assumptions (3.34) and (3.35), one has to find a divergence free \mathbf{h} such that the components of \mathbf{h} are independent of the \mathbf{e}_a for the Fefferman-Graham equations to become linear.

Remark 3.2. Note that the assumptions of Theorem 3.1 imply that $\overset{0}{\nabla}_X \mathbf{e}_a \in \mathcal{K}$ but not that \mathcal{K} or $\mathcal{K}^\perp = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_p)$ are parallel distributions. Indeed, the terms

$$2\overset{0}{\mathbf{g}}(\overset{0}{\nabla}_i \mathbf{e}_a, \mathbf{e}_A) = \overset{0}{g}([\mathbf{e}_i, \mathbf{e}_a], \mathbf{e}_A) + \overset{0}{g}([\mathbf{e}_A, \mathbf{e}_a], \mathbf{e}_i) + \overset{0}{g}([\mathbf{e}_A, \mathbf{e}_i], \mathbf{e}_a)$$

might be non-zero for $i = B$ or $i = \bar{c}$.

Remark 3.3. In order to obtain that all $\mathbf{h}(\rho, \cdot)$ are divergence free, in additions to the conditions on the frame in Theorem 3.1 we can impose an additional condition

$$(3.36) \quad [\mathbf{e}_A, \mathbf{e}_a] \in \mathcal{K}^\perp.$$

Indeed, we have

$$\operatorname{div}(\mathbf{h})(\mathbf{e}_i) = -g^{a\bar{c}}\mathbf{h}(\mathbf{e}_{\bar{c}}, \overset{0}{\nabla}_{\mathbf{e}_a}\mathbf{e}_i) - g^{AB}\mathbf{h}(\overset{0}{\nabla}_{\mathbf{e}_B}\mathbf{e}_A, \mathbf{e}_i) = -g^{AB}\mathbf{h}(\overset{0}{\nabla}_{\mathbf{e}_B}\mathbf{e}_A, \mathbf{e}_i).$$

The remaining term $\mathbf{h}(\overset{0}{\nabla}_{\mathbf{e}_B}\mathbf{e}_A, X)$ vanishes if $\overset{0}{\nabla}_{\mathbf{e}_B}\mathbf{e}_A \in \mathcal{K}$. We get

$$2\mathbf{h}(\overset{0}{\nabla}_{\mathbf{e}_B}\mathbf{e}_A, \mathbf{e}_a) = \overset{0}{\mathbf{g}}([\mathbf{e}_a, \mathbf{e}_A], \mathbf{e}_B) + \overset{0}{\mathbf{g}}([\mathbf{e}_a, \mathbf{e}_B], \mathbf{e}_A),$$

which is zero because of the assumption (3.36).

4. AMBIENT METRICS FOR NULL RICCI WALKER METRICS

In this section we apply the results of the previous section to conformal classes given by a null Ricci Walker metric \mathbf{g} as defined in the introduction. First we review some results about Walker metrics, then focus on the Ricci tensor, and finally draw the conclusions from the previous sections about the ambient metric of null Ricci Walker-manifolds. Note that in Sections 4.1 and 4.2 we drop the suffix 0 on $\overset{0}{\mathbf{g}}$ for brevity, and use it again in Section 4.3 when we need to distinguish between $\overset{0}{\mathbf{g}}$ and the ρ -dependent family \mathbf{g} .

4.1. Walker manifolds. A pseudo-Riemannian manifold (M, \mathbf{g}) is a *Walker manifold* if there is vector distribution $\mathcal{N} \subset TM$ of rank $p > 0$ that is a totally null with respect to \mathbf{g} and invariant under parallel transport with respect to the Levi-Civita connection of \mathbf{g} . The most comprehensive study of Walker manifolds can be found in [2]. In the following we will derive a description that is useful for our purpose and allows us to construct examples.

Proposition 4.1. *Let (M, \mathbf{g}) be a pseudo-Riemannian manifold of dimension n . Then the following conditions are equivalent*

- (1) (M, \mathbf{g}) is a Walker manifold with parallel null distribution \mathcal{N} .
- (2) There exists local coordinates (x^1, \dots, x^n) , so-called Walker coordinates, such that

$$\mathbf{g} = 2dx^{\bar{a}}(\delta_{\bar{a}b}dx^b + F_{\bar{a}B}dx^B + H_{\bar{a}\bar{b}}dx^{\bar{b}}) + G_{AB}dx^A dx^B,$$

where the $F_{\bar{a}B}$ and G_{AB} independent of the x^a 's. Here we use the same index conventions as in (3.2) as well as $\delta_{\bar{a}b} = 1$ if $\bar{a} = n - p + b$ and zero otherwise. In these coordinates, the parallel null distribution \mathcal{N} is given by the span of the $\partial_a = \frac{\partial}{\partial x^a}$'s.

- (3) There is a frame $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ with dual frame $(\Theta^1, \dots, \Theta^n)$ such that

$$\mathbf{g} = 2g_{a\bar{c}}\Theta^a \circ \Theta^{\bar{c}} + g_{AB}\Theta^A \circ \Theta^B,$$

with constants $g_{a\bar{c}}$ and g_{AB} and such that

$$(4.1) \quad \begin{aligned} \mathcal{K} &= \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_{n-p}) \text{ is involutive,} \\ [\mathbf{e}_a, \mathbf{e}_b] &= [\mathbf{e}_a, \mathbf{e}_B] = 0 \\ [\mathbf{e}_a, \mathbf{e}_{\bar{c}}] &\in \mathcal{K}^\perp \text{ and } [\mathbf{e}_B, \mathbf{e}_{\bar{c}}] \in \mathcal{K}. \end{aligned}$$

In this frame $\mathcal{N} = \mathcal{K}^\perp = \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_p)$.

Proof. The equivalence of items (1) and (2) is due to Walker [16]. In order to show that (2) implies (3), we fix some Walker coordinates (x^1, \dots, x^n) such that

$$\mathbf{g} = 2dx^{\bar{a}}(\delta_{\bar{a}\bar{b}}dx^{\bar{b}} + F_{\bar{a}B}dx^B + H_{\bar{a}\bar{b}}dx^{\bar{b}}) + G_{AB}dx^A dx^B$$

with $F_{\bar{a}B}$ and G_{AB} independent of the x^a 's. Then we set

$$\mathbf{e}_a := \partial_a, \quad \mathbf{e}_A := C_A^B \left(\partial_B - F_{\bar{a}B} \delta^{\bar{a}\bar{b}} \partial_{\bar{b}} \right), \quad \mathbf{e}_{\bar{c}} := \partial_{\bar{c}} - H_{\bar{a}\bar{c}} \delta^{\bar{a}\bar{b}} \partial_{\bar{b}},$$

where C_A^B is a matrix such that $C_A^B G_{BE} C_D^E = \delta_{AD}$. Note that, since G_{AB} does not depend on the x^a 's, also C_A^B does not depend on the x^a 's. We claim that this frame satisfies all the conditions (4.1). Clearly, the metric in this frame has the right form and $[\mathbf{e}_a, \mathbf{e}_{\bar{b}}] = 0$. But also the other commutator relations are satisfied:

$$\begin{aligned} [\mathbf{e}_a, \mathbf{e}_{\bar{c}}] &= \left[\partial_a, \partial_{\bar{c}} - H_{\bar{a}\bar{c}} \delta^{\bar{a}\bar{b}} \partial_{\bar{b}} \right] = -dH_{\bar{a}\bar{c}}(\partial_a) \delta^{\bar{a}\bar{b}} \partial_{\bar{b}} \in \mathcal{K}^\perp \\ [\mathbf{e}_a, \mathbf{e}_A] &= \left[\partial_a, C_A^B \left(\partial_B - F_{\bar{c}B} \delta^{\bar{c}\bar{d}} \partial_{\bar{d}} \right) \right] = 0 \\ [\mathbf{e}_{\bar{c}}, \mathbf{e}_A] &= \left[\partial_{\bar{c}} - H_{\bar{a}\bar{c}} \delta^{\bar{a}\bar{b}} \partial_{\bar{b}}, C_A^B \left(\partial_B - F_{\bar{c}B} \delta^{\bar{c}\bar{d}} \partial_{\bar{d}} \right) \right] \in \mathcal{K} \end{aligned}$$

This shows that all the conditions (4.1) are satisfied.

Conversely, we have to show that the bracket relations (4.1) imply that $\nabla_X \mathbf{e}_a \in \mathcal{N} = \mathcal{K}^\perp$. For this we use the Koszul formula

$$2\mathbf{g}(\nabla_{\mathbf{e}_i} \mathbf{e}_a, \mathbf{e}_j) = \mathbf{g}([\mathbf{e}_i, \mathbf{e}_a], \mathbf{e}_j) + \mathbf{g}([\mathbf{e}_j, \mathbf{e}_a], \mathbf{e}_i) + \mathbf{g}([\mathbf{e}_j, \mathbf{e}_i], \mathbf{e}_a).$$

From (4.1) it follows that this is zero for all $j = a$ and $j = B$. Hence, $\nabla_X \mathbf{e}_a \in \mathcal{N} = \mathcal{K}^\perp = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_p)$. \square

Next we record formulas for the curvature of a Walker metric.

Lemma 4.1. *Let (M, \mathbf{g}) be a Walker manifold and let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be a frame as in (3) of Proposition 4.1 such that \mathbf{g} is given as in (3.4).*

- (1) *Let Γ_{ij}^k the connection components with respect to the frame $(\mathbf{e}_1, \dots, \mathbf{e}_n)$, i.e., defined by $\nabla_i \mathbf{e}_j = \Gamma_{ij}^k \mathbf{e}_k$. Then*

$$\begin{aligned} \Gamma_{ab}^k &= \Gamma_{ba}^k = \Gamma_{Ab}^k = \Gamma_{bA}^k = 0 \\ \Gamma_{ai}^B &= \Gamma_{ia}^B = \Gamma_{a\bar{i}}^{\bar{B}} = \Gamma_{\bar{i}a}^{\bar{B}} = \Gamma_{Ai}^{\bar{B}} = \Gamma_{iA}^{\bar{B}} = 0 \end{aligned}$$

- (2) *The curvature tensor and the Ricci tensor of \mathbf{g} satisfy*

$$R_{ijaB} = 0, \quad R_{ab} = R_{aB} = 0$$

for all $a, b = 1, \dots, p$, $B = p+1, \dots, n-p$ and $i, j = 1, \dots, n$.

Proof. The properties of the connection components are a direct consequence of \mathcal{K} and \mathcal{K}^\perp being parallel distributions and of the Koszul formula

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\mathbf{g}([\mathbf{e}_i, \mathbf{e}_j], \mathbf{e}_l) + \mathbf{g}([\mathbf{e}_l, \mathbf{e}_j], \mathbf{e}_i) + \mathbf{g}([\mathbf{e}_l, \mathbf{e}_i], \mathbf{e}_j)).$$

As \mathcal{K} and \mathcal{K}^\perp are parallel distributions, in the given frame, the curvature tensor of a Walker manifold satisfies

$$R_{ijab} = R_{ijAb} = 0$$

Indeed, we have for example

$$R_{biAd} = \mathbf{g}(\overset{0}{R}(\mathbf{e}_b, \mathbf{e}_i)\mathbf{e}_A, \mathbf{e}_d) = 0$$

since \mathcal{K} is parallel and thus $R(\mathbf{e}_b, \mathbf{e}_i)\mathbf{e}_A \in \mathcal{K}$. This implies that the components of the Ricci tensor

$$R_{ai} = g^{b\bar{c}}(R_{bai\bar{c}} + R_{\bar{c}aib}) + g^{AB}R_{AaiB} = g^{b\bar{c}}R_{\bar{c}aib}$$

are zero unless $i = \bar{d}$. \square

4.2. Null Ricci Walker metrics. Here we study the Ricci tensor of a Walker metric. The results will provide a method of constructing examples of Walker metrics with two step nilpotent Ricci tensor in Section 5, in particular for the examples of Lie groups with left-invariant metric.

Proposition 4.2. *Let \mathbf{g} be a metric as in (3.4) and assume that the frame $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ satisfies the following bracket relations*

$$[\mathbf{e}_i, \mathbf{e}_j] = r_{ij}^k \mathbf{e}_k$$

with smooth functions r_{ij}^k . If we assume that

$$(4.2) \quad r_{ab}^k = r_{aB}^k = r_{a\bar{c}}^{\bar{b}} = r_{a\bar{c}}^B = r_{B\bar{c}}^{\bar{a}} = 0$$

(these are just the conditions in Proposition 4.1) and that

$$(4.3) \quad r_{AB}^C = r_{AB}^{\bar{c}} = 0,$$

with

$$(4.4) \quad dr_{b\bar{c}}^d(\mathbf{e}_a) = dr_{b\bar{c}}^{\bar{d}}(\mathbf{e}_a) = dr_{b\bar{c}}^d(\mathbf{e}_A) = dr_{b\bar{c}}^{\bar{d}}(\mathbf{e}_A) = 0,$$

$$(4.5) \quad dr_{BC}^d(\mathbf{e}_A) = dr_{B\bar{c}}^D(\mathbf{e}_A) = 0,$$

then \mathbf{g} is a Walker metric whose curvature satisfies in addition

$$R_{aijb} = R_{ABCi} = R_{abD\bar{c}} = 0.$$

In particular, \mathbf{g} null Ricci Walker, i.e.,

$$R_{ai} = R_{Ai} = 0.$$

Proof. First we prove the vanishing of the curvature components R_{bijd} . Because of the previous lemma we only have to show that $R_{b\bar{a}\bar{c}d} = 0$. In terms of the r_{ij}^k 's the connection coefficients Γ_{ij}^k write as

$$(4.6) \quad \Gamma_{ij}^k = \frac{1}{2}r_{ij}^k + g^{kl}r_{l(i}g_{j)m} = \frac{1}{2}r_{ij}^k - g^{kl}g_{m(i}r_{j)l}^m.$$

After imposing the condition on the frame to define a Walker metric, i.e., after imposing equations (4.2), Lemma 4.1 leaves us with connection coefficients $\Gamma_{a\bar{c}}^b$, Γ_{AB}^b , Γ_{AB}^C , $\Gamma_{A\bar{c}}^b$, $\Gamma_{A\bar{c}}^B$ and $\Gamma_{a\bar{c}}^k$. Imposing the additional condition (4.3), $r_{AB}^C = 0$, implies

$$\Gamma_{AB}^C = -g^{CD}g_{E(A}r_{B)D}^E = 0.$$

Because of $\Gamma_{ab}^k = 0$ and $[\mathbf{e}_a, \mathbf{e}_{\bar{c}}] = r_{a\bar{c}}^b \mathbf{e}_b$ we obtain

$$R_{b\bar{a}d\bar{c}} = \mathbf{g}(\nabla_b \nabla_{\bar{a}} \mathbf{e}_d, \mathbf{e}_{\bar{c}}) = \left(d\Gamma_{\bar{a}d}^f(\mathbf{e}_b) + \Gamma_{\bar{a}d}^k \Gamma_{bk}^f \right) g_{f\bar{c}} = d\Gamma_{\bar{a}d}^f(\mathbf{e}_b) g_{f\bar{c}}$$

by Lemma 4.1. Using (4.6) we get

$$\Gamma_{\bar{a}d}^f = \frac{1}{2} \left(r_{\bar{a}d}^f - g^{f\bar{c}} (g_{e\bar{a}} r_{d\bar{c}}^e + g_{\bar{e}d} dr_{\bar{a}\bar{c}}^{\bar{e}}) \right)$$

and the assumption that $dr_{b\bar{c}}^d(\mathbf{e}_a) = dr_{\bar{a}\bar{c}}^{\bar{b}}(\mathbf{e}_a) = 0$ yields that $d\Gamma_{\bar{a}d}^f(\mathbf{e}_b)$ and hence $R_{b\bar{a}d\bar{c}}$ vanishes. It also implies that

$$R_{\bar{a}b} = g^{\bar{c}d} R_{d\bar{a}b\bar{c}} = 0.$$

It remains to verify that $R_{Ai} = 0$. We start with

$$R_{AB} = g^{CD} R_{CABD}.$$

Computing R_{CABD} we get on the one hand

$$\mathbf{g}(\nabla_{[\mathbf{e}_C, \mathbf{e}_A]} \mathbf{e}_B, \mathbf{e}_D) = r_{CA}^a \Gamma_{aB}^E g_{ED} = 0,$$

because of Lemma 4.1. On the other hand, because of (4.3) we have $\Gamma_{AB}^E = 0$ and hence

$$\mathbf{g}(\nabla_C \nabla_A \mathbf{e}_B, \mathbf{e}_D) = (\partial_C(\Gamma_{AB}^E) + \Gamma_{AB}^k \Gamma_{Ck}^E) g_{ED} = 0$$

since $\Gamma_{AB}^E = \Gamma_{Ca}^E = \Gamma_{AB}^{\bar{c}} = 0$. This implies that $g^{CD} R_{CABD} = 0$ as well as $R_{AB} = 0$. Finally, we compute

$$R_{B\bar{a}} = g^{b\bar{c}} R_{\bar{c}B\bar{a}b} + g^{AC} R_{AB\bar{a}C}$$

First we look at the term $R_{ABC\bar{a}}$ and compute

$$\mathbf{g}(\nabla_{[\mathbf{e}_A, \mathbf{e}_B]} \mathbf{e}_C, \mathbf{e}_{\bar{a}}) = r_{AB}^b \Gamma_{bC}^d g_{d\bar{a}} = 0,$$

as $\Gamma_{bC}^d = 0$. On the other hand we have

$$\mathbf{g}(\nabla_A \nabla_B \mathbf{e}_C, \mathbf{e}_{\bar{a}}) = \left(\mathbf{e}_A(\Gamma_{BC}^d) + \Gamma_{BC}^k \Gamma_{Ak}^d \right) g_{d\bar{a}} = g_{d\bar{a}} \mathbf{e}_A(\Gamma_{BC}^d)$$

as $\Gamma_{BC}^D = \Gamma_{BC}^{\bar{c}} = 0$. Now

$$\Gamma_{BC}^d = \frac{1}{2} r_{BC}^d - g^{b\bar{c}} g_{C(A} r_{B)}^C{}_{\bar{c}}$$

and because of (4.5), $dr_{BC}^d(\mathbf{e}_A) = dr_{B\bar{c}}^D(\mathbf{e}_A) = 0$, we get $\partial_A(\Gamma_{BC}^d) = 0$. Hence, we get $R_{ABC\bar{a}} = 0$ and thus $R_{ABCi} = 0$. For the Ricci component $R_{B\bar{a}} = g^{b\bar{c}} R_{\bar{c}B\bar{a}b}$ we compute

$$\begin{aligned} R_{\bar{c}B\bar{a}b} &= \left(\partial_{\bar{c}}(\Gamma_{Bb}^d) - \partial_B(\Gamma_{\bar{c}b}^d) + \Gamma_{Bb}^k \Gamma_{\bar{c}k}^d - \Gamma_{\bar{c}b}^k \Gamma_{Bk}^d - r_{\bar{c}B}^k \Gamma_{kb}^d \right) g_{d\bar{a}} \\ &= \left(-\partial_B(\Gamma_{\bar{c}b}^d) - \Gamma_{\bar{c}b}^k \Gamma_{Bk}^d - r_{\bar{c}B}^k \Gamma_{kb}^d \right) g_{d\bar{a}} \end{aligned}$$

by Lemma 4.1 which also tells us that $\Gamma_{Bb}^k = 0$ and that

$$\Gamma_{\bar{c}b}^k \Gamma_{Bk}^d = \Gamma_{\bar{c}b}^a \Gamma_{Ba}^d = 0.$$

Assumption (4.3) implies that

$$r_{\bar{c}B}^k \Gamma_{kb}^d = r_{\bar{c}B}^f \Gamma_{fb}^d + r_{\bar{c}B}^D \Gamma_{Db}^d = 0.$$

Furthermore, we have

$$\Gamma_{\bar{c}b}^d = \frac{1}{2} \left(r_{\bar{c}b}^d - g^{d\bar{a}} (g_{e\bar{c}} r_{b\bar{a}}^e + g_{\bar{e}b} r_{\bar{c}\bar{a}}^{\bar{e}}) \right)$$

Moreover, the assumptions (4.4) imply that $\partial_B(\Gamma_{\bar{c}b}^d) = 0$ and finally we get that $R_{\bar{c}B\bar{a}b} = 0$ and that $R_{B\bar{a}}$ which yields that \mathbf{g} has two step nilpotent Ricci tensor. \square

Remark 4.1. In view of the examples we will construct in Section 5, note that in general the remaining Ricci components do not vanish, even if all the r_{ij}^k 's are constant:

$$\begin{aligned} R_{\bar{a}\bar{c}} &= 2g^{b\bar{d}}R_{b(\bar{a}\bar{c})\bar{d}} + g^{BD}R_{B(\bar{a}\bar{c})D} \\ &= 2 \left(d\Gamma_{(\bar{a}\bar{c})}^b(\mathbf{e}_b) - d\Gamma_{b(\bar{a})}^b(\mathbf{e}_{\bar{c}}) + \Gamma_{b(\bar{a})}^d r_{\bar{c})d}^b - \Gamma_{b(\bar{a})}^d \Gamma_{\bar{c})d}^b + \Gamma_{bd}^b \Gamma_{(\bar{a}\bar{c})}^{\bar{d}} \right. \\ &\quad \left. + d\Gamma_{(\bar{a}\bar{c})}^A(\mathbf{e}_A) - d\Gamma_{A(\bar{a})}^A(\mathbf{e}_{\bar{c}}) + \Gamma_{B(\bar{a})}^A r_{\bar{c})A}^B + \Gamma_{B(\bar{a})}^A \Gamma_{\bar{c})A}^B + \Gamma_{(\bar{a}\bar{c})}^{\bar{d}} \Gamma_{A\bar{d}}^A \right). \end{aligned}$$

4.3. The Fefferman-Graham equations for null Ricci Walker metrics. Here we apply our results of Theorems 3.1 and 2.1 to null Ricci Walker metrics. The following theorem will imply Theorem 1.3 and consequently Theorem 1.1 from the introduction.

Theorem 4.1. *Let $(M, \overset{0}{\mathbf{g}})$ be a null Ricci Walker-manifold with parallel totally null distribution \mathcal{N} such that $\text{Im}(\mathbf{P}) \subset \mathcal{N}$. Then an ambient metric $\tilde{\mathbf{g}} = 2dtd(\rho t) + t^2\mathbf{g}(\rho)$ for $[\overset{0}{\mathbf{g}}]$ in the sense of Definition 2.1 is given by $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$, where $\mathbf{h} = \mathbf{h}(\rho)$ is divergence free bilinear form with $\text{Im}(\mathbf{h}) \subset \mathcal{N}$ that satisfies the PDE*

$$(4.7) \quad \rho \ddot{h}_{ij} - \frac{n-2}{2} \dot{h}_{ij} - \frac{1}{2} \square h_{ij} + \overset{0}{R}_{kijl} h^{kl} + \overset{0}{R}_{ij} + \frac{1}{2} \left(h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij} + \overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}^l h^k_j \right) = O(\rho^m),$$

for $m = \infty$ if n is odd and $m = \frac{n-2}{2}$ when n is even. Here $\mathbf{h} = (h_{ij})$, $\overset{0}{R}_{ijkl}$ denotes the curvature tensor, $\overset{0}{R}_{ij}$ the Ricci tensor and $\square h_{ij} = \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij}$, all with respect to $\overset{0}{\mathbf{g}}$.

Proof. Let $\tilde{\mathbf{g}} = 2dtd(\rho t) + t^2\mathbf{g}(\rho)$ be an ambient metric for the conformal class of $\overset{0}{\mathbf{g}}$ in the sense of Definition 2.1. Then, from Theorem 2.1 we know that there is a $\mathbf{h} = \mathbf{g} - \overset{0}{\mathbf{g}}$ that is divergence free and its image is contained in \mathcal{N} . Then \mathbf{h} and $\mathcal{K} = \mathcal{N}^\perp$ satisfy the assumptions of Theorem 3.1. Hence, the term quadratic in \mathbf{h} in the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ is given by equation (3.24). Note that, since \mathcal{K} is parallel, the second term in (3.24) simplifies to

$$\overset{0}{\nabla}_{[k} h_{l]i} \overset{0}{\nabla}^k h_j^l = -\frac{1}{2} \overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}^l h^k_j.$$

Moreover, since $\text{Im}(\mathbf{P}) \subset \mathcal{N}$ and $\text{Im}(\mathbf{h}) \subset \mathcal{N}$, in (3.33) the product of \mathbf{h} with the Ricci tensor of $\overset{0}{g}$ vanishes,

$$\overset{0}{R}^k_i h_{jk} = \frac{1}{n-2} \mathbf{P}^k_i h_{jk} = 0.$$

The other direction follows immediately from Theorem 3.1 as \mathbf{h} and $\mathcal{K} = \mathcal{N}^\perp = \text{Im}(\mathbf{P})^\perp$ for a null Ricci Walker-manifold satisfy its assumptions. \square

We are now going to fix a basis $\mathbf{e}_a = \partial_a, \mathbf{e}_B, \mathbf{e}_{\bar{c}}$ as in Proposition 4.1 with dual basis $\Theta^a, \Theta^B, \Theta^{\bar{c}}$ so that $\mathbf{h} = h_{\bar{a}\bar{c}} \Theta^{\bar{a}} \Theta^{\bar{c}}$ and write the equation (4.7) as a PDE system for the functions $h_{\bar{a}\bar{c}}$. Note that all terms in (4.7) are zero unless both indices i, j are barred, i.e. $i = \bar{b}$ and $j = \bar{d}$.

To this end note that for a Walker metric the properties listed in Proposition 4.1 imply that

$$(4.8) \quad \overset{0}{\nabla}_b \Theta^{\bar{a}} = \overset{0}{\nabla}_A \Theta^{\bar{a}} = 0.$$

It is

$$(4.9) \quad \overset{0}{\nabla} \mathbf{h} = dh_{\bar{a}\bar{c}} \otimes (\Theta^{\bar{c}} \cdot \Theta^{\bar{c}}) + 2h_{\bar{a}\bar{c}} (\overset{0}{\nabla} \Theta^{\bar{a}} \cdot \Theta^{\bar{c}})$$

and hence

$$(4.10) \quad \begin{aligned} \overset{0}{\nabla}_X \overset{0}{\nabla}_Y \mathbf{h} &= (\overset{0}{\nabla}_X dh_{\bar{b}\bar{d}})(Y) \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + 2h_{\bar{b}\bar{d}} \left(\overset{0}{\nabla}_X \overset{0}{\nabla}_Y \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + \overset{0}{\nabla}_X \Theta^{\bar{b}} \cdot \overset{0}{\nabla}_Y \Theta^{\bar{d}} \right) \\ &\quad + 2dh_{\bar{b}\bar{d}}(X) \nabla_Y \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + 2dh_{\bar{b}\bar{d}}(Y) \nabla_X \Theta^{\bar{b}} \cdot \Theta^{\bar{d}}. \end{aligned}$$

First we consider the quadratic terms in (4.7). Because of (4.8), for $h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij}$ we get

$$h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{\bar{b}\bar{d}} = h^{cd} \overset{0}{\nabla}_c \overset{0}{\nabla}_d h_{\bar{b}\bar{d}} = h^{cd} \partial_c \partial_d (h_{\bar{b}\bar{d}}).$$

Similarly, the term $\overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}^l h^k_j$ is computed as

$$\overset{0}{\nabla}_k h_{l\bar{b}} \overset{0}{\nabla}^l h^k_{\bar{d}} = \overset{0}{g}^{\bar{a}\bar{e}} \overset{0}{g}^{\bar{c}f} \mathbf{e}_b(h_{\bar{c}\bar{d}} \mathbf{e}_f(h_{\bar{e}\bar{d}})) = \partial_a(h^c_{\bar{b}}) \partial_c(h^a_{\bar{d}}).$$

Next, we study the linear terms in (4.7). For the $\square \mathbf{h}$ -term in (4.7) we get

$$\begin{aligned} \square \mathbf{h} &= \overset{0}{\nabla}^k \overset{0}{\nabla}_k \mathbf{h} = \overset{0}{g}^{a\bar{c}} \left(\overset{0}{\nabla}_a \overset{0}{\nabla}_{\bar{c}} \mathbf{h} + \overset{0}{\nabla}_{\bar{c}} \overset{0}{\nabla}_a \mathbf{h} \right) + \overset{0}{g}^{AB} \overset{0}{\nabla}_A \overset{0}{\nabla}_B \mathbf{h} \\ &= \square(h_{\bar{b}\bar{d}}) \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + 2h_{\bar{b}\bar{d}} \square \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + 4\overset{0}{g}^{a\bar{c}} dh_{\bar{b}\bar{d}}(\mathbf{e}_a) \overset{0}{\nabla}_{\bar{c}} \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} \end{aligned}$$

because of (4.9). The term $\square \Theta^{\bar{b}}$ we can express using the curvature

$$\square \Theta^{\bar{b}} = \overset{0}{g}^{a\bar{c}} \overset{0}{\nabla}_a \overset{0}{\nabla}_{\bar{c}} \Theta^{\bar{b}} = \overset{0}{g}^{a\bar{c}} \overset{0}{R}_{a\bar{c}}^{\bar{b}} \Theta^{\bar{c}} = \overset{0}{g}^{a\bar{c}} \overset{0}{R}_{a\bar{c}}^{\bar{b}} \Theta^{\bar{c}}.$$

Moreover, since $\overset{0}{\nabla}_{\bar{c}} \Theta^{\bar{d}}|_{\mathcal{K}} = 0$ we may define connection coefficients $\Gamma^{a\bar{b}}_{\bar{c}}$ by

$$\overset{0}{g}^{a\bar{c}} \overset{0}{\nabla}_{\bar{c}} \Theta^{\bar{b}} = \Gamma^{a\bar{b}}_{\bar{c}} \Theta^{\bar{c}}$$

to obtain

$$\square \mathbf{h} = \square(h_{\bar{b}\bar{d}}) \Theta^{\bar{b}} \cdot \Theta^{\bar{d}} + 2h_{\bar{b}\bar{d}} \overset{0}{g}^{a\bar{c}} \overset{0}{R}_{a\bar{c}}^{\bar{b}} \Theta^{\bar{c}} \cdot \Theta^{\bar{d}} + 4dh_{\bar{b}\bar{d}}(\mathbf{e}_a) \Gamma^{a\bar{b}}_{\bar{c}} \Theta^{\bar{c}} \cdot \Theta^{\bar{d}}.$$

This can be written as

$$(4.11) \quad \square h_{\bar{b}\bar{d}} = \square(h_{\bar{b}\bar{d}}) + 2\overset{0}{g}^{a\bar{c}} \overset{0}{R}_{a\bar{c}}^{\bar{b}} \Theta^{\bar{c}} h_{\bar{d}} + 4\partial_a(h_{\bar{c}\bar{d}}) \Gamma^{a\bar{c}}_{\bar{b}}.$$

Altogether we obtain

Proposition 4.3. *Let $(M, \overset{0}{\mathbf{g}})$ be a null Ricci Walker-manifold with a fixed frame $\mathbf{e}_a = \partial_a, \mathbf{e}_B, \mathbf{e}_{\bar{c}}$ with a dual frame $\Theta^a, \Theta^B, \Theta^{\bar{c}}$ as in Proposition 4.1. Then equation (4.7) for $\mathbf{h} = h_{\bar{b}\bar{d}} \Theta^{\bar{b}} \Theta^{\bar{d}}$ in Theorem 4.1 is equivalent to the following PDE system for the components $h_{\bar{b}\bar{d}}$ of \mathbf{h} :*

$$(4.12) \quad \begin{aligned} \rho \ddot{h}_{\bar{b}\bar{d}} - \frac{n-2}{2} \dot{h}_{\bar{b}\bar{d}} - \frac{1}{2} \square(h_{\bar{b}\bar{d}}) - \overset{0}{g}^{a\bar{c}} \overset{0}{R}_{a\bar{c}}^{\bar{b}} \Theta^{\bar{c}} h_{\bar{d}} + \overset{0}{R}_{\bar{b}\bar{d}}^{\bar{a}} \Theta^{\bar{a}} h_{\bar{a}\bar{c}} + \overset{0}{R}_{\bar{b}\bar{d}} \\ - 2\partial_a(h_{\bar{c}\bar{d}}) \Gamma^{a\bar{c}}_{\bar{b}} + \frac{1}{2} (h^{cd} \partial_c \partial_d (h_{\bar{b}\bar{d}}) + \partial_a(h^c_{\bar{b}}) \partial_c(h^a_{\bar{d}})) = 0, \end{aligned}$$

where $\overset{0}{R}_{ijkl}$ denotes the curvature tensor, $\overset{0}{R}_{ij}$ the Ricci tensor with respect to $\overset{0}{\mathbf{g}}$.

This proposition shows for a null Ricci Walker metric, that the terms in the Fefferman-Graham equations that are non-linear in \mathbf{h} vanish whenever the components $h_{\bar{b}\bar{d}}$ of \mathbf{h} do not depend on the coordinates x^a in Proposition 4.1 corresponding to the total null plane, i.e., if

$$\mathcal{L}_{\partial_a} \mathbf{h}_{\bar{b}\bar{d}} = \partial_a (h_{\bar{b}\bar{d}}) = 0.$$

In the following we will present two situations in which this assumption is satisfied.

4.4. Null Ricci Walker metrics with parallel null line. In the case when the parallel null distribution has rank one, i.e. $p = 1$ and $\mathcal{N} = \mathbb{R} \cdot \mathbf{e}_1$, the property $\mathcal{L}_{\mathbf{e}_1} \mathbf{h} = 0$ is implied by $\mathbf{h} = h(\Theta^n)^2$ being divergence free. Indeed, we have

$$\operatorname{div}(\mathbf{h}) = \overset{0}{\nabla}_k h^k_i = \mathcal{L}_{\mathbf{e}_1} \mathbf{h} = \partial_1(h).$$

Hence we obtain one part of Corollary 1.1 in the introduction, in which the Fefferman-Graham equation reduce to linear equations:

Corollary 4.1. *Let $(M, \overset{0}{\mathbf{g}})$ be a null Ricci Walker-manifold with a parallel null line \mathcal{N} and $\operatorname{Im}(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{\mathbf{g}} = 2\operatorname{dtd}(\rho t) + t^2 \mathbf{g}(\rho)$ for $\overset{0}{\mathbf{g}}$ is an ambient metric in the sense of Definition 2.1 if and only if there is a tensor \mathbf{h} such that $\mathbf{h} = \mathbf{h}(\rho) = \mathbf{g}(\rho) - \overset{0}{\mathbf{g}}$ that satisfies $\operatorname{Im}(\mathbf{h}) \subset \mathcal{N}$, $\mathcal{L}_X \mathbf{h} = 0$ for all $X \in \mathcal{N}$ and solves the linear PDE*

$$(4.13) \quad \rho \ddot{h}_{ij} - \frac{n-2}{2} \dot{h}_{ij} - \frac{1}{2} \overset{0}{\square} h_{ij} + \overset{0}{R}_{kijl} h^{kl} + \overset{0}{R}_{ij} = O(\rho^m),$$

for $m = \infty$ if n is odd and $m = \frac{n-2}{2}$ if n is even. Here $\mathbf{h} = (h_{ij})$, $\overset{0}{R}_{ijkl}$ is the curvature tensor, $\overset{0}{R}_{ij}$ the Ricci tensor and $\overset{0}{\square} h_{ij} = \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij}$, all with respect to $\overset{0}{\mathbf{g}}$.

More precisely, in a frame $\mathbf{e}_1 = \partial_1, \mathbf{e}_B, \mathbf{e}_n$ such that $\mathcal{N} = \mathbb{R} \cdot \mathbf{e}_1$ and with a dual frame $\Theta^1, \Theta^B, \Theta^n$ as in Proposition 4.1, $\tilde{\mathbf{g}}$ is Ricci-flat if and only if $\mathbf{h} = h(\Theta^n)^2$ for a function h with $\partial_1(h) = 0$ satisfying the following linear PDE

$$(4.14) \quad \rho \ddot{h} - \frac{n-2}{2} \dot{h} - \frac{1}{2} \overset{0}{\square}(h) + 2\overset{0}{R}_{1nn1} h + \overset{0}{R}_{nn} = 0.$$

4.5. Another class of null Ricci Walker metrics with linear Fefferman-Graham equations. Returning to the general case $p \geq 1$, in order for the Fefferman-Graham equations to reduce to linear equations, we have seen that the property $\mathcal{L}_{\mathbf{e}_a}(\mathbf{h}) = 0$ is crucial. Here we present another class for which this property is satisfied. First we prove a result that strengthens Theorem 1.2 for this class:

Proposition 4.4. *Let $\overset{0}{\mathbf{g}}$ be a null Ricci Walker metric with parallel null distribution \mathcal{N} and Schouten tensor \mathbf{P} . Assume furthermore that*

$$(4.15) \quad X \lrcorner \overset{0}{R} = 0, \quad \text{for all } X \in \mathcal{N},$$

where $\overset{0}{R}$ is the curvature tensor of $\overset{0}{\mathbf{g}}$. Then $\mathcal{L}_X \mathbf{P} = 0$ for all $X \in \mathcal{N}$.

Moreover, let $\tilde{\mathbf{g}} = 2\operatorname{dtd}(\rho t) + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$ be an ambient metric for $\overset{0}{\mathbf{g}}$ in the sense of Definition 2.1. Then, in addition to (2.9) and (2.10) in Theorem 2.1, it holds that

$$(4.16) \quad \mathcal{L}_X \mathbf{h} = O(\rho^m), \quad \text{for all } X \in \mathcal{N},$$

for all m if n is odd and for $m \leq \frac{n}{2} - 1$ if n is even. When n is even, in addition to the properties in Theorem 2.1, the obstruction tensor satisfies

$$\mathcal{L}_X \mathcal{O} = 0, \quad \text{for all } X \in \mathcal{N},$$

and one can choose an ambient metric such that the corresponding \mathbf{h} satisfies (4.16) for all $m \geq 1$.

Proof. The Bianchi-identity applied to $\nabla_a \overset{0}{R}_{ijkl}$ is used to show that (4.15) implies that $\overset{0}{\nabla}_a \mathbf{P}_{ij} = 0$. But for a null Ricci Walker metrics this is equivalent to $\mathcal{L}_{\mathbf{e}_a} \mathbf{P} = 0$.

The statement about $\mathbf{h} = \sum_{m \geq 1} \frac{1}{m!} \overset{m}{h} \rho^m$ in the ambient metric is proved in a similar way by induction over m as in the proof of Theorem 2.1. But now the computations are simplified, as we can use equations (4.7) in Theorem 4.1, which are equivalent to the Fefferman-Graham equations:

Applying $\overset{0}{\nabla}_a$ to equation (4.7), differentiating it $(m-1)$ times with respect to ρ , for $m \leq \frac{n}{2} - 1$ when n is even, and using the induction hypothesis yields

$$0 = (m - \frac{n}{2}) \overset{0}{\nabla}_a \overset{m}{h}_{ij} - \frac{1}{2} \overset{0}{g}^{kl} \overset{0}{\nabla}_a \overset{0}{\nabla}_k \overset{0}{\nabla}_l \overset{m-1}{h}_{ij} + \overset{0}{\nabla}_a \overset{0}{R}_{kijl} \overset{m-1}{h}^{kl} = (m - \frac{n}{2}) \overset{0}{\nabla}_a \overset{m}{h}_{ij},$$

Here we use the Bianchi identity and that (4.15) allows to commute $\overset{0}{\nabla}_a$ with $\overset{0}{\nabla}_k$. This equation shows (4.16) and when $m = \frac{n}{2}$ the result for the obstruction tensor.

Finally, when n is even, the terms $\overset{m}{h}$ for $m \geq \frac{n}{2}$ are not determined by the Fefferman-Graham equations. So we can choose them in a way that $\mathcal{L}_{\mathbf{e}_a} \overset{m}{h}_{ij} = \partial_a(h_{ij}) = 0$. \square

A tensor \mathbf{h} satisfying (4.16) for all m is automatically divergence free

$$\overset{0}{\nabla}_k h^k_{\bar{c}} = \overset{0}{g}^{\bar{a}b} (\overset{0}{\nabla}_b h_{\bar{a}\bar{c}} + \overset{0}{\nabla}_{\bar{a}} h_{b\bar{c}}) + \overset{0}{g}^{AB} \overset{0}{\nabla}_A h_{B\bar{c}} = \overset{0}{g}^{\bar{a}b} \partial_b(h_{\bar{a}\bar{c}}) = 0.$$

Moreover, for such a \mathbf{h} the quadratic terms in (4.7) vanish since $\overset{0}{\nabla}_a h_{ij} = 0$, and the term $\overset{0}{\square}(\mathbf{h}_{\bar{b}\bar{c}})$, i.e. the wave operator of $\overset{0}{\mathbf{g}}$ applied to the components $h_{\bar{b}\bar{d}}$ in (4.14), simplifies to

$$\overset{0}{\Delta}(h_{\bar{b}\bar{d}}) = \overset{0}{g}^{AC} \overset{0}{\nabla}_A \overset{0}{\nabla}_C(h_{\bar{b}\bar{d}}),$$

which is the wave operator for the metric $g_{AC} \Theta^A \Theta^C$ in $n-2p$ dimensions. Finally, the vanishing of the curvature terms $\overset{0}{R}_{aijk}$ implies that the system (4.14), in addition to becoming linear, decouples to $\frac{p+1}{2}$ single equations on the $\frac{p+1}{2}$ components $h_{\bar{b}\bar{d}}$. These equation only differ in their inhomogeneity:

Corollary 4.2. *Let $(M, \overset{0}{\mathbf{g}})$ be a null Ricci Walker-manifold with parallel totally null distribution \mathcal{N} and $\text{Im}(\mathbf{P}) \subset \mathcal{N}$ and such that that*

$$X \lrcorner \overset{0}{R} = 0, \quad \text{for all } X \in \mathcal{N},$$

where $\overset{0}{R}$ is the curvature tensor of $\overset{0}{\mathbf{g}}$. Then, the metric

$$\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$$

is an ambient metric for $[\mathbf{g}]^0$ if and only if, each of the components $h_{\bar{b}\bar{d}}$ of \mathbf{h} in a basis as in Proposition 4.1 satisfy the following inhomogeneous linear PDE

$$(4.17) \quad \Delta_-(h_{\bar{b}\bar{c}}) + 2R_{\bar{b}\bar{d}}^0 = O(\rho^m),$$

where $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even and where Δ_- is the linear second order differential operator defined by

$$(4.18) \quad \Delta_-(f) = 2\rho\ddot{f} + (2-n)\dot{f} - \overset{0}{\Delta}(f)$$

for the function $f = f(x^{p+1}, \dots, x^n, \rho)$ and with $\overset{0}{\Delta}(f) = \overset{0}{g}^{AC}\overset{0}{\nabla}_A\overset{0}{\nabla}_C(f) = \overset{0}{g}^{AC}\mathbf{e}_A(\mathbf{e}_C(f))$.

This corollary and Proposition 4.4 imply the remaining statements in Corollary 1.1. Note that for null Ricci Walker metrics we have that $\mathcal{L}_X\mathcal{O} = \nabla_X\mathcal{O} = 0$ for all $X \in \mathcal{N}$. A construction method for metrics satisfying the assumptions is provided by Proposition 4.2. Explicit examples will be constructed in the next section.

Remark 4.2. Note that the equality $\overset{0}{g}^{AB}\overset{0}{\nabla}_A\overset{0}{\nabla}_B(h_{\bar{a}\bar{c}}) = \overset{0}{g}^{AB}\mathbf{e}_A(\mathbf{e}_B(h_{\bar{a}\bar{c}}))$ follows from $\overset{0}{\nabla}_A\mathbf{e}_B = \Gamma_{AB}^d\mathbf{e}_d$ and $dh_{\bar{a}\bar{c}}(\mathbf{e}_d) = 0$.

Note also that an integrability condition for each of the equations (4.17) is $dR_{\bar{b}\bar{d}}(\mathbf{e}_a) = 0$. This condition is satisfied if, in addition, we assume that

$$dr_{ij}^k(\mathbf{e}_a) = 0, \quad \text{for all } i, j, k = 1, \dots, n,$$

as in Proposition 4.2.

5. EXAMPLES WITH EXPLICIT AMBIENT METRICS

In this section we will provide examples of conformal classes of null Ricci Walker metrics for which we find explicit solutions to equation (4.17) obtaining explicit examples of Ricci-flat ambient metrics.

5.1. Solving the homogeneous equation. Equation (4.17) is a linear, inhomogeneous PDE for each of the functions $h_{\bar{a}\bar{c}}$ given by the linear differential operator

$$\Delta_- = 2\rho\partial_\rho^2 + (2-n)\partial_\rho - \overset{0}{\Delta}.$$

In the section we will find metrics for which we get an explicit solution of (4.17). Before this, we start by providing the solution to the homogeneous equation.

Lemma 5.1. *Let M be a smooth manifold of dimension n and \mathcal{D} some linear differential operator on M . For a function $F \in C^\infty(M)$ we define the functions $F_\pm \in C^\infty(M \times (-\epsilon, \epsilon))$ as*

$$F_\pm := \sum_{k=1}^{\infty} \frac{\mathcal{D}^k(F)}{k! \prod_{i=1}^k (2i \pm n)} \rho^k,$$

where F_- is only defined when n is odd or $\mathcal{D}^{\frac{n}{2}}(F) = 0$. Moreover, define the following linear differential operators on $C^\infty(M \times (-\epsilon, \epsilon))$

$$\mathcal{D}_\pm := 2\rho\partial_\rho^2 + (2 \pm n)\partial_\rho - \mathcal{D}.$$

Then, for any $F \in C^\infty(M)$ and $f \in C^\infty(M \times (-\epsilon, \epsilon))$ we have

$$(5.1) \quad \mathcal{D}_\pm(F_\pm) = \mathcal{D}(F)$$

$$(5.2) \quad \mathcal{D}_-(\rho^{\frac{n}{2}} f) = \rho^{\frac{n}{2}} \mathcal{D}_+(f).$$

$$(5.3) \quad \mathcal{D}_-(\rho^{\frac{n}{2}} F_+) = \rho^{\frac{n}{2}} \mathcal{D}_+(F_+) = \rho^{\frac{n}{2}} \mathcal{D}(F).$$

$$(5.4) \quad \mathcal{D}_-(\rho^{\frac{n}{2}}(F + F_+)) = 0.$$

In particular, for each $F \in C^\infty(M)$, the function $f = \rho^{\frac{n}{2}}(F + F_+)$ is a solution to the homogeneous equation $\mathcal{D}_-(f) = 0$.

Proof. To verify equations (5.1) and (5.2) is a straightforward computation. Both together imply (5.3) which yields (5.4). \square

5.2. Extensions of nilpotent Lie algebras. Let \mathfrak{k} be a two-step nilpotent Lie algebra of dimension q and let \mathfrak{z} be its centre of dimension $p < q$. We fix a complement \mathfrak{m} of \mathfrak{z} ,

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{m}$$

Then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$ and we can fix a basis $(\mathbf{e}_a)_{a=1, \dots, p}$ of \mathfrak{z} and $(\mathbf{e}_A)_{A=p+1, \dots, q}$ of \mathfrak{m} such that

$$[\mathbf{e}_a, \mathbf{e}_b] = 0, \quad [\mathbf{e}_a, \mathbf{e}_B] = 0, \quad [\mathbf{e}_A, \mathbf{e}_B] = r_{AB}^C \mathbf{e}_C,$$

where r_{AB}^C denote the structure constants of \mathfrak{k} . Note that there are no further conditions on these numbers other than $r_{AB}^C = -r_{BA}^C$. Denote by $\mathfrak{der}(\mathfrak{k})$ the derivations of \mathfrak{k} which comes with a canonical Lie algebra structure induced from $\mathfrak{gl}(\mathfrak{k})$. Note that derivations leave the centre invariant.

Furthermore, let H be a Lie group with Lie algebra \mathfrak{h} and of dimension $p = \dim(\mathfrak{z})$ and $\phi : \mathfrak{h} \rightarrow \mathfrak{der}(\mathfrak{k})$ a Lie algebra homomorphism from \mathfrak{h} to the derivations of \mathfrak{k} . By fixing a basis $(\mathbf{e}_{\bar{a}})_{\bar{a}=q+1, \dots, p+q}$ of \mathfrak{h} , we can write ϕ as

$$\phi(\mathbf{e}_{\bar{a}})\mathbf{e}_b = r_{b\bar{a}}^d \mathbf{e}_d, \quad \phi(\mathbf{e}_{\bar{a}})\mathbf{e}_B = r_{B\bar{a}}^d \mathbf{e}_d + r_{B\bar{a}}^E \mathbf{e}_E,$$

with some constants $r_{b\bar{a}}^d$, $r_{B\bar{a}}^d$ and $r_{B\bar{a}}^E$. Finally, with respect to this basis denote the structure constants of \mathfrak{h} by $r_{\bar{a}\bar{b}}^{\bar{c}}$, i.e.,

$$[\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{b}}] = r_{\bar{a}\bar{b}}^{\bar{c}} \mathbf{e}_{\bar{c}}.$$

Now we define the Lie algebra \mathfrak{g} to be semi-direct sum $\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{k}$ of \mathfrak{h} and \mathfrak{k} with respect to ϕ of dimension $n = p + q$. Clearly, the structure constants of \mathfrak{g} are given by the numbers

$$r_{AB}^C, r_{b\bar{a}}^d, r_{B\bar{a}}^d, r_{B\bar{a}}^E, r_{\bar{a}\bar{b}}^{\bar{c}},$$

which are subject to the conditions $r_{ij}^k = -r_{ji}^k$ and

$$r_{AB}^E r_{\bar{c}\bar{c}}^d = -2r_{\bar{c}[A}^C r_{B]C}^d,$$

i.e., that $\phi(\mathbf{e}_{\bar{c}})$ is a derivation, as well as

$$r_{\bar{a}\bar{b}}^{\bar{c}} r_{d\bar{c}}^e = 2r_{d[\bar{a}}^c r_{\bar{b}]c}^e, \quad r_{\bar{a}\bar{b}}^{\bar{c}} r_{A\bar{c}}^d = 2r_{A[\bar{a}}^c r_{\bar{b}]c}^d + 2r_{A[\bar{a}}^B r_{\bar{b}]B}^d, \quad r_{\bar{a}\bar{b}}^{\bar{c}} r_{A\bar{c}}^B = 2r_{A[\bar{a}}^C r_{\bar{b}]C}^B,$$

which ensure that $\phi : \mathfrak{k} \rightarrow \mathfrak{der}(\mathfrak{h})$ is a Lie algebra homomorphism. The frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ on the Lie group G corresponding to \mathfrak{g} satisfies the bracket relations of Proposition 4.2 with the parallel distribution \mathcal{K} given by \mathfrak{k} . Now we define a left invariant metric by formula (3.4)

$$\mathbf{g} = g_{a\bar{c}}(\Theta^a \otimes \Theta^{\bar{c}} + \Theta^{\bar{c}} \otimes \Theta^a) + \overset{0}{g}_{AB} \Theta^A \circ \Theta^B$$

where the Θ^i 's are again the algebraic duals of the \mathbf{e}_i 's and the g_{ij} are constants. Now the distribution \mathcal{K}^\perp is given by \mathfrak{z} . Then Proposition 4.2 implies that (G, \mathbf{g}) is a null Ricci Walker manifold of dimension n , which, in general is not Ricci flat. Its possibly non vanishing components are given by constants $R_{\bar{a}\bar{c}}$.

In order to determine the ambient metric for the conformal class given by \mathbf{g} on G , we have to solve equations (4.17) in this setting, i.e., find a functions $h \in C^\infty((-\varepsilon, \varepsilon) \times G)$, such that

$$(5.5) \quad 2\rho\ddot{h} + (2-n)\dot{h} - \Delta(h) + C = 0, \quad \text{with initial condition } h|_{\rho=0} \equiv 0,$$

with $\Delta(h) = g^{AB}\nabla_A\nabla_B h$, and for constants C that are given by the components of the Ricci tensor $R_{\bar{a}\bar{c}}$. Equation (5.5), when taken along $\rho = 0$ implies

$$\dot{h}|_{\rho=0} \equiv \frac{C}{n-2}.$$

Clearly, the problem (5.5) has a linear solution

$$h = \frac{C}{n-2}\rho,$$

but Lemma 5.1 shows that there are more solutions. From Corollary 4.2 we obtain Theorem 1.4 from the introduction. More precisely, we get

Theorem 5.1. *Let \mathfrak{k} be a two-step nilpotent Lie algebra of dimension q with centre \mathfrak{z} of dimension $p \leq q$, and let H be a Lie group of dimension p and with Lie algebra \mathfrak{h} . Let $\phi : \mathfrak{h} \rightarrow \mathfrak{der}(\mathfrak{k})$ a Lie algebra homomorphism into the derivations of \mathfrak{k} and G be the $n = q + p$ -dimensional Lie group corresponding to the Lie algebra \mathfrak{g} that is given as the semi-direct sum*

$$\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{k},$$

of \mathfrak{h} and \mathfrak{k} by ϕ . Fix a basis $(\mathbf{e}_{\bar{a}})_{\bar{a}=1,\dots,p}$ of \mathfrak{h} , a basis $(\mathbf{e}_a)_{a=1,\dots,p}$ of \mathfrak{z} and complement it with $(\mathbf{e}_A)_{A=1,\dots,q-p}$ to a basis of \mathfrak{k} . Let $(\Theta^i)_{i=1,\dots,n}$ is the dual basis to $(\mathbf{e}_i)_{i=1,\dots,n}$ and

$$\mathbf{g} = 2g_{\bar{a}\bar{c}}\Theta^{\bar{a}} \circ \Theta^{\bar{c}} + g_{AB}\Theta^A \circ \Theta^B$$

be the left-invariant pseudo-Riemannian metric \mathbf{g} on G defined by real numbers $g_{\bar{a}\bar{c}}$ and g_{AB} . Then the conformal class of \mathbf{g} on G admits Ricci-flat ambient metrics given by

$$\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2 \left(\mathbf{g} + \left(\frac{2\rho}{n-2} R_{\bar{a}\bar{c}} + \rho^{\frac{n}{2}} \left(F_{\bar{a}\bar{c}} + \sum_{k=1}^{\infty} \frac{\overset{0}{\Delta}^k(F_{\bar{a}\bar{c}})}{k! \prod_{i=1}^k (2i+n)} \rho^k \right) \right) \Theta^{\bar{a}} \Theta^{\bar{c}} \right),$$

where $R_{\bar{a}\bar{c}} = \text{Ric}^{\mathbf{g}}(\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{c}})$ are the components of the Ricci tensor of \mathbf{g} and $F_{\bar{a}\bar{c}} = F_{\bar{c}\bar{a}}$ are functions on G with $dF_{\bar{a}\bar{c}}(\mathbf{e}_a) = 0$. In particular, when n is odd, $F_{\bar{a}\bar{c}} \equiv 0$ gives the unique analytic Ricci-flat Fefferman-Graham ambient metric.

Note that in general the metrics \mathbf{g} as in the theorem are neither Ricci flat nor do they admit parallel null vector fields (see also Remark 4.1).

5.3. Generalised pp-waves. Another class of examples to which our Corollary 4.2 applies are the Lorentzian pp-waves for which we have determined the analytic ambient metric in [12]. The acronym “pp” stands for *plane fronted with parallel rays*. A Lorentzian pp-wave metric in dimension n is locally given by

$$\mathbf{g} = 2dudv + Hdu^2 + \sum_{i=1}^{n-2} (dx^i)^2,$$

where $H = H(x^i, u)$ is a function that does not depend on v .

Here we generalise this class and the results in [1, 12] to higher signature and, more importantly, determine all solutions to the Fefferman-Graham equations including the non-analytic ones, and determine the obstruction tensor in the case of Lorentzian pp-waves.

We will use the same index conventions as in the previous sections ($a = 1, \dots, p$, $B = p + 1, \dots, n - p$, $\bar{c} = n - p + 1, \dots, n$), and define a modified Kronecker delta as

$$\delta_{\bar{a}b} = \begin{cases} 1, & \text{if } \bar{a} = b + n - p \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.1. Let $\mathcal{U} \subset \mathbb{R}^n \ni (x^1, \dots, x^n)$ be an open set, and $H_{\bar{a}\bar{c}}$ and G_{AB} smooth functions on \mathcal{U} satisfying $\det(G_{AB}) \neq 0$ and $\partial_a(H_{\bar{a}\bar{c}}) = 0$ and $\partial_a(G_{AB}) = \partial_{\bar{c}}(G_{AB}) = 0$. Then the pseudo-Riemannian metric

$$(5.6) \quad \mathbf{g} = 2\delta_{\bar{a}b}dx^{\bar{a}}dx^b + H_{\bar{a}\bar{b}}dx^{\bar{a}}dx^{\bar{b}} + G_{AB}dx^A dx^B,$$

is called a *generalised pp-wave*, or for short, a *gpp-wave*.

If all the G_{AB} ’s are constants, we call \mathbf{g} *plane fronted wave with parallel rays*, or for short, *pp-wave*.

To obtain Lorentzian gpp-waves, one sets $p = 1$ and G_{AB} positive definite. For all p , gpp-waves admit p parallel vector fields ∂_a and hence are Walker metrics, however in general not null Ricci Walker metrics. As in Proposition 4.1, for gpp-waves we have the frame and dual co-frame

$$\begin{aligned} \mathbf{e}_a &:= \partial_a, & \mathbf{e}_B &:= E_B^A \partial_A, & \mathbf{e}_{\bar{c}} &:= \partial_{\bar{c}} - H_{\bar{a}\bar{c}} \delta^{\bar{a}\bar{b}} \partial_b, \\ \Theta^a &= dx^a + H_{\bar{a}\bar{c}} dx^{\bar{c}}, & \Theta^B &= F_A^B dx^A, & \Theta^{\bar{c}} &= dx^{\bar{c}}, \end{aligned}$$

where E_A^B is a matrix such that $E_A^B G_{BC} E_D^C = \delta_{AD}$ and F_A^B is the inverse of E_A^B . Note that, since G_{AB} does not depend on the x^a ’s or the $x^{\bar{c}}$ ’s neither does E_A^B . The gpp-wave metric in this frame is

$$\mathbf{g} = \delta_{\bar{a}\bar{b}} \Theta^a \Theta^{\bar{b}} + g_{AB} \Theta^A \Theta^B.$$

with $g_{AB} = \epsilon_A \delta_{AB}$. The only non vanishing brackets for this frame are

$$\begin{aligned} [\mathbf{e}_A, \mathbf{e}_B] &= -E_{[A}^C E_{B]}^D dF_D^E (\partial_C) \mathbf{e}_E, \\ [\mathbf{e}_A, \mathbf{e}_{\bar{b}}] &= -dH_{\bar{b}\bar{c}}(\mathbf{e}_A) \delta^{\bar{c}\bar{d}} \mathbf{e}_{\bar{d}} \\ [\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{b}}] &= 2dH_{\bar{c}[\bar{a}}(\partial_{\bar{b}}) \delta^{\bar{c}\bar{d}} \mathbf{e}_{\bar{d}}, \end{aligned}$$

Hence, the assumptions of Proposition 4.2 are satisfied whenever the G_{AB} ’s are constant, i.e., whenever \mathbf{g} is a pp-wave.

The Levi-Civita connection ∇ of a gpp-wave \mathbf{g} is given by

$$\begin{aligned}\nabla_A \mathbf{e}_B &= \nabla_A^{\mathbf{G}} \mathbf{e}_B, \\ \nabla_{\bar{a}} \mathbf{e}_B &= dH_{\bar{a}\bar{c}}(\mathbf{e}_B) \delta^{\bar{c}\bar{b}} \mathbf{e}_b, \\ \nabla_{\bar{a}} \mathbf{e}_{\bar{b}} &= -2dH_{\bar{a}[\bar{b}}(\partial_{\bar{c}]}) \delta^{\bar{c}\bar{d}} \mathbf{e}_d - \text{grad}^{\mathbf{G}}(H_{\bar{a}\bar{b}}),\end{aligned}$$

in which $\nabla^{\mathbf{G}}$ is the Levi-Civita connection of the metric $\mathbf{G} = G_{AB}dx^A dx^B$ and $\text{grad}^{\mathbf{G}}$ the corresponding gradient. This allows us to compute the curvature, which satisfies $R_{aijk} = 0$, and the Ricci-curvature, whose only possibly non-vanishing terms are given as

$$\begin{aligned}R_{AB} &= R_{AB}^{\mathbf{G}} \\ R_{\bar{a}\bar{c}} &= -\frac{1}{2}g^{BD}\mathbf{g}(\nabla_B(\text{grad}(H_{\bar{a}\bar{c}}), \mathbf{e}_D) = -\frac{1}{2}g^{BD}\nabla_B^{\mathbf{G}}\nabla_D^{\mathbf{G}}(H_{\bar{a}\bar{c}}) = -\frac{1}{2}\Delta_{\mathbf{G}}(H_{\bar{a}\bar{c}}).\end{aligned}$$

Lemma 5.2. *The defined gpp-waves satisfy $R_{aijk} = 0$ and they are null Ricci Walker metrics if the metric \mathbf{G} is Ricci-flat. In particular, pp-waves are null Ricci Walker metrics.*

Remark 5.1. If we drop the assumption on a pp-wave that the \mathbf{e}_a 's are parallel, i.e., that $\partial_a H \neq 0$, then the Ricci tensor is no longer two-step nilpotent. For example in the Lorentzian case, i.e., when $p = 1$ and $\epsilon_i = 1$, if $\partial_1 H \neq 0$ we get that

$$\text{Ric}(\partial_1, \partial_n) = \partial_1^2(H), \quad \text{Ric}(\partial_A, \partial_n) = \partial_A \partial_1(H),$$

which shows that Ric cannot be two-step nilpotent (see also [10]).

Remark 5.2. Using the necessary conditions that were derived in [6] for conformal Einstein metrics, a straightforward computation of the Weyl, Cotton and Bach tensors as in [12] shows that in general gpp-waves are not conformally Einstein. In fact, in [12] we gave explicit examples of Bach flat pp-waves that are not conformally Einstein.

When determining the ambient metric for a gpp-wave for which the metric \mathbf{G} is Ricci-flat, we can apply Theorem 4.1 and Proposition 4.4. Moreover, since all the $\mathbf{e}_a = \partial_a$ are parallel, the curvature terms R_{aijk} vanish, but also the $\Theta^{\bar{a}}$'s are parallel. We obtain

Corollary 5.1. *Let $\mathbf{G} = G_{AB}dx^A dx^B$ be a Ricci-flat metric on \mathbb{R}^{n-2p} and $H_{\bar{a}\bar{b}}$ functions of $(n-p)$ variables $(x^A, x^{\bar{b}})$ that define the gpp-wave*

$$\mathbf{g} = 2\delta_{\bar{a}\bar{b}}dx^{\bar{a}}dx^{\bar{b}} + H_{\bar{a}\bar{b}}dx^{\bar{a}}dx^{\bar{b}} + G_{AB}dx^A dx^B$$

on \mathbb{R}^n . Then an ambient metric for $[\mathbf{g}]$ is given by $\tilde{\mathbf{g}} = 2dtd(\rho t) + t^2(\mathbf{g} + \mathbf{h}(\rho))$, where $\mathbf{h} = h_{\bar{b}\bar{d}}dx^{\bar{b}}dx^{\bar{d}}$ and whose components satisfy $\partial_a(h_{\bar{b}\bar{d}}) = 0$ and

$$(5.7) \quad 2\rho\ddot{h}_{\bar{b}\bar{d}} + (2-n)\dot{h}h_{\bar{b}\bar{d}} - \Delta_{\mathbf{G}}(h_{\bar{b}\bar{d}}) - \Delta_{\mathbf{G}}(H_{\bar{b}\bar{d}}) = O(\rho^m),$$

with $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even and where $\Delta_{\mathbf{G}}$ is the Laplacian of \mathbf{G} .

This corollary shows that in order to obtain Ricci flat ambient metrics, for a function $H = H(x^{p+1}, \dots, x^n)$ we have to solve the equation

$$(5.8) \quad 2\rho\ddot{h} + (2-n)\dot{h} - \Delta_{\mathbf{G}}(h) - \Delta_{\mathbf{G}}(H) = 0,$$

for a function $h = h(\rho, x^{p+1}, \dots, x^n)$. This can be solved by standard power series expansion, noticing that its indicial exponents are $s = 0$ and $s = n/2$. We extend our results in [1, 12], by the following more general existence statement for gpp-waves.

Theorem 5.2. *Let \mathbf{G} be a semi-Riemannian metric on \mathbb{R}^{n-2p} . Then the following functions $h = h(\rho, x^{p+1}, \dots, x^n)$ are solutions to equation (5.8) with $h(\rho) \rightarrow 0$ when $\rho \downarrow 0$:*

When n is odd:

$$(5.9) \quad h = \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i - n)} \rho^k + \rho^{n/2} \left(\alpha + \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k \alpha}{k! \prod_{i=1}^k (2i + n)} \rho^k \right),$$

where $\alpha = \alpha(x^{p+1}, \dots, x^n)$ is an arbitrary function of its variables. In particular, if $\alpha \equiv 0$ this gives an analytic in ρ solution in a neighbourhood of $\rho = 0$ with $h(0) = 0$.

When $n = 2s$ is even:

$$(5.10) \quad h = \sum_{k=1}^{s-1} \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i - n)} \rho^k + \rho^s \left(\alpha + \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k \alpha}{k! \prod_{i=1}^k (2i + n)} \rho^k \right) + c_n \rho^s \left(\sum_{k=0}^{\infty} (\log(\rho) - q_k) \frac{\Delta_{\mathbf{G}}^{s+k} H}{k! \prod_{i=1}^k (2i + n)} \rho^k \right),$$

where $\alpha = \alpha(x^{p+1}, \dots, x^n)$ and $q_0 = q_0(x^{n-p+1}, \dots, x^n)$ and

$$q_k(x^{n-p+1}, \dots, x^n) := q_0(x^{n-p+1}, \dots, x^n) + \sum_{i=1}^k \frac{n + 4i}{i(n + 2i)},$$

for $k = 1, 2, \dots$, are arbitrary functions of their variables and the constant c_n is given as follows

$$c_n := -\frac{1}{(s-1)! \prod_{i=0}^{s-1} (2i - n)}.$$

In particular, when $\Delta_{\mathbf{G}}^s H \equiv 0$ there are solutions that are analytic in ρ in a neighbourhood of $\rho = 0$ and with $h(0) = 0$. These solutions are parametrized by the functions α .

Proof. That the given function satisfy equation (5.8) can be checked directly. In the case n odd it follows from Lemma 5.1. For n even, the situation is a bit more subtle. We give the formulas for each term, ignoring the term $(\rho^{\frac{n}{2}}(\alpha + \alpha_+))$, for which we have seen that it is in the kernel of \mathcal{D}_- :

$$\begin{aligned} \mathcal{D}_- \left(\sum_{k=1}^{s-1} \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i - n)} \rho^k \right) &= \Delta_{\mathbf{G}} H - \frac{\Delta_{\mathbf{G}}^s H}{(s-1)! \prod_{i=1}^{s-1} (2i - n)} \rho^{s-1}, \\ \mathcal{D}_- (\rho^s \Delta_{\mathbf{G}}^s (\log(\rho)(H + H_+))) &= n \rho^{s-1} \Delta_{\mathbf{G}}^s H + \frac{n+4}{n+2} \rho^s \Delta_{\mathbf{G}}^{s+1} H \\ &\quad + \sum_{k=1}^{\infty} \frac{(n+4(k+1))}{(k+1)! \prod_{i=1}^{k+1} (2i + n)} \Delta_{\mathbf{G}}^{s+k+1} H \rho^{s+k} \\ \mathcal{D}_- \left(\rho^s \Delta_{\mathbf{G}}^s \sum_{k=0}^{\infty} q_k \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i - n)} \rho^k \right) &= (q_1 - q_0) \rho^s \Delta_{\mathbf{G}}^{s+1} H, \\ &\quad + \sum_{k=1}^{\infty} \frac{(q_{k+1} - q_k)(n+2(k+1))}{k! \prod_{i=1}^{k+1} (2i + n)} \Delta_{\mathbf{G}}^{s+k+1} H \rho^{s+k}. \end{aligned}$$

Looking at the ρ^{s-1} -terms in these formulas we determine c_n as in the theorem by

$$-\frac{1}{(s-1)!\prod_{i=1}^{s-1}(2i-n)} + nc_n = 0.$$

Moreover, looking at the ρ^s -terms, we determine q_1 by

$$\frac{n+4}{n+2} - (q_1 - q_0) = 0$$

as given in the theorem, and finally the other q_k 's by

$$n + 4(k+1) - (q_{k+1} - q_k)(n + 2(k+1))(k+1) = 0.$$

This proves the theorem. \square

Summarising, we obtain

Corollary 5.2. *Let*

$$\mathbf{g} = 2dx^{\bar{a}}(\delta_{\bar{a}\bar{b}}dx^{\bar{b}} + H_{\bar{a}\bar{b}}dx^{\bar{b}}) + G_{AB}dx^A dx^B$$

be a gpp-wave with Ricci flat metric $\mathbf{G} = G_{AB}dx^A dx^B$. Then ambient metrics in the sense of Definition 2.1 for the conformal class $[\mathbf{g}]$ are

$$\begin{aligned} \tilde{\mathbf{g}} = & 2d(\rho t)dt + t^2\mathbf{g} + \\ & + t^2 \left(\left(\sum_{k=1}^m \frac{\Delta_{\mathbf{G}}^k(H_{\bar{a}\bar{b}})}{k! \prod_{i=1}^k (2i-n)} \rho^k + \rho^{n/2} \left(F_{\bar{a}\bar{b}} + \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k(F_{\bar{a}\bar{b}})}{k! \prod_{i=1}^k (2i+n)} \rho^k \right) \right) dx^{\bar{a}} dx^{\bar{b}} \right) \end{aligned}$$

in which $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even, and $F_{\bar{a}\bar{c}} = F_{\bar{c}\bar{a}}$ are arbitrary functions on M , with $\partial_a(F_{\bar{a}\bar{c}}) = 0$. Moreover,

- (1) *When n is odd, $F_{\bar{a}\bar{c}} \equiv 0$ gives the unique analytic Ricci-flat Fefferman-Graham ambient metric.*
- (2) *When n is even and $\Delta_{\mathbf{G}}^{\frac{n}{2}}(H_{\bar{a}\bar{c}}) = 0$, the metric $\tilde{\mathbf{g}}$ is Ricci flat.*
- (3) *When n is even and $\Delta_{\mathbf{G}}^{\frac{n}{2}}(H_{\bar{a}\bar{c}}) \neq 0$, Ricci-flat but non analytic ambient metrics are given by formula (5.10).*

5.4. Ambient metrics for Lorentzian pp-waves. Finally we consider Lorentzian pp-waves, i.e., gpp-waves with $p = 1$ and $G_{AB} = \delta_{AB}$. Since $p = 1$ we use a different convention as names for the variables: we replace coordinates $x^1, x^A, A = 2, \dots, n-2$, and x^n by $v := x^1, y^i = x^{i+1}, i = 1, \dots, n-2$, and $u = x^n$. We have seen solutions of equation (5.8) in Theorem 5.2. For Lorentzian pp-waves these are all of the solutions. Here $\Delta_{\mathbf{G}} = \Delta$ is just the flat Laplacian and we can use the Fourier transform to transform equation (5.8) into an ODE. In fact, in [1] we proved the following

Theorem 5.3 ([1]). *Let Δ be the flat Laplacian in $(n-2)$ dimensions.*

When n is odd, the most general solutions h to equation (5.8) with $h(\rho) \rightarrow 0$ when $\rho \downarrow 0$ are given by formula (5.9) in Theorem 5.2 and parametrized by arbitrary functions $\alpha = \alpha(x^1, \dots, x^{n-2}, u)$. In particular, there is a unique solution that is analytic in ρ in a neighbourhood of $\rho = 0$ with $h(0) = 0$. This solution is given by $\alpha \equiv 0$.

When $n = 2s$ is even, the most general solutions h to equation (5.8) with $h(\rho) \rightarrow 0$ when $\rho \downarrow 0$ are given by

$$(5.11) \quad h = \sum_{k=1}^{s-1} \frac{\Delta^k H}{k! \prod_{i=1}^k (2i - n)} \rho^k + \rho^s \left(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k! \prod_{i=1}^k (2i + n)} \rho^k \right) \\ + c_n \rho^s \sum_{k=0}^{\infty} \frac{1}{k! \prod_{i=1}^k (2i + n)} \left((\log(\rho) - q_k) \Delta^{s+k} H + Q * \Delta^{s+k} H \right) \rho^k,$$

where $\alpha = \alpha(y^i, u)$ and $Q = Q(x^i, u)$ are arbitrary functions of their variables, $*$ denotes the convolution of two functions with respect to the y^i -variables, c_n is the constant defined in Theorem 5.2, and the other constants are given as follows

$$q_0 := 0, \quad q_k := \sum_{i=1}^k \frac{n + 4i}{i(n + 2i)}, \quad \text{for } k = 1, 2, \dots$$

In particular, only when $\Delta^s H \equiv 0$ there are solutions that are analytic in ρ in a neighbourhood of $\rho = 0$ and with $h(0) = 0$. These solutions are not unique but parametrized by the functions α .

As a conclusion, for Lorentzian pp-waves we get the complete picture in Theorem 1.5:

Corollary 5.3. *Let*

$$(5.12) \quad \mathbf{g} = 2du dv + H du^2 + \sum_{i=1}^{n-2} (dy^i)^2$$

be a Lorentzian pp-wave metric with $H = H(y^1, \dots, y^{n-2}, u)$ a function not depending on v . Let Δ is the flat Laplacian in $n - 2$ dimensions.

- (1) *If n is odd, the unique Ricci-flat ambient metric that is analytic in ρ is*

$$\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2 \mathbf{g} + t^2 \left(\sum_{k=1}^{\infty} \frac{\Delta^k(H)}{k! \prod_{i=1}^k (2i - n)} \rho^k \right) du^2.$$

Moreover, all non-analytic solutions are parametrized by arbitrary functions $\alpha = \alpha(y^1, \dots, y^{n-2}, u)$ and given by formula (5.9) in Theorem 5.2, in which $\Delta_{\mathbf{g}}$ is replaced by the flat Laplacian.

- (2) *If $n = 2s$ is even the obstruction tensor for $[\mathbf{g}]$ is a constant multiple of $\Delta^{n/2}(H)du^2$. If it vanishes, all Ricci flat ambient metrics that are analytic in ρ are given by*

$$\tilde{\mathbf{g}} = 2d(\rho t)dt + t^2 \mathbf{g} + t^2 \left(\sum_{k=1}^{s-1} \frac{\Delta^k(H)}{k! \prod_{i=1}^k (2i - n)} \rho^k + \sum_{k=0}^{\infty} \frac{\Delta^k(\alpha)}{k! \prod_{i=1}^k (2i + n)} \rho^{\frac{n}{2}+k} \right) du^2,$$

where $\alpha = \alpha(y^1, \dots, y^{n-2}, u)$ is an arbitrary smooth function. Independently of the vanishing of the obstruction tensor, non-analytic ambient metrics can be obtained from formula (5.11) in Theorem 5.3.

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